Tutorial 4

Projectile Motion - Elementary Analysis

Contents

4.1 A Simple Experiment

One can perform some conceptually simple experiments to show that objects tend to fall directly toward the earth with an approximate constant acceleration (defined g) of 9.81 m/s^2 or 32.2 ft/s^2.

Using only a stop watch and a long measuring tape as experimental tools, several students dropped a small metal ball from different height buildings. The students measured the drop time and drop distance three times, averaging the results for each building. Assuming that the object falls at constant acceleration, they calculated the downward acceleration of the ball at each building.

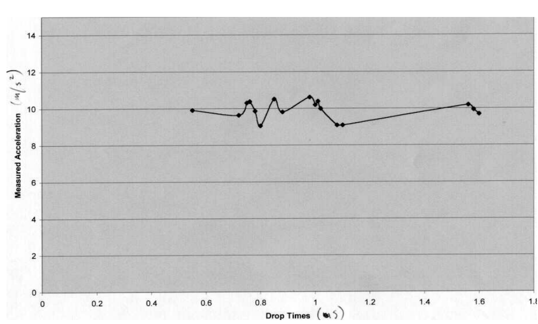

Figure 4.1: Drop acceleration for different drop times

As can be seen from the data (to within experimental error, which is considerable), the metal ball fell at slightly less than $10 m/s^2$.

It is important to note that a small metal ball was used. This is true, since small dense objects tend to be less restrained by air resistance. Air resistance is one of the effects that we ignore here. The analyses herein contain a host of other approximations and these are discussed in the next section.

4.2 Projectile Motion Approximations

Projectile motion is considered here with the following approximations:

1. **Ignore Earth Curvature and Structure**
   - (a) Projectile velocity small enough that curvature of earth can be ignored.
   - (b) Projectile velocity small enough that spinning of earth can be ignored.
   - (c) Ignore obstacles.
2. **Constant Gravity/Empty Universe**
   - (a) Projectile velocity small enough that variable earth gravity can be ignored. Thus, the initial projectile velocity is small compared to the escape velocity.
   - (b) Gravitational attractions from other planets and objects (terrestrial and extraterrestrial) are ignored.
   - (c) Ignore wobble of earth (from projectile) during projectile path.
   - (d) No extraterrestrial objects strike the projectile.
   - (e) No sunlight or any other form of energy impart significant momentum to the projectile.
3. **Simple Projectile**
   - (a) Projectile has no net charge, so that radiation from the particle can be ignored.
   - (b) The projectile doesn't change structurally, i.e. it doesn't spontaneously blow up, decay into something else, or change spatially (e.g. person releasing a parachute)
4. ***Neglect Air Resistance***
   - (a) Negligible air resistance so that projectiles don't lose energy.
   - (b) Negligible air resistance/slow spinning projectiles so that projectile does not wobble or "curve".
   - (c) Negligible wind resistance.
   - (d) Projectile velocity small enough that it does not reach the atmosphere, i.e. ignore variable air resistance.
   - (e) Projectile velocity small compared to the speed of sound, so that it does not lose energy creating a sonic boom.
5. **Ignore Relativistic and Quantum Effects**
   - (a) Projectile velocity slow compared to the speed of light, so that relativistic effects can be ignored.
   - (b) DeBroglie wavelength small compared to the size of the projectile, so that quantum mechanical effects can be ignored.
   - (c) Electron-positron pairs don't pop up and change the path of the projectile. (i.e. Ignore Quantum Vacuum Fluctuations)

4.3 Constant Gravity Equations

If y is measured straight "up", and x is measured along the ground, then it follows that

$$a_x = 0 \tag{4.1}$$

$$a_y = -g \tag{4.2}$$

Integrating these yields

$$v_x = v_{x0} \tag{4.3}$$

$$v_y = v_{y0} - gt \tag{4.4}$$

Integrating these and applying the initial conditions yields

$$x = x_0 + v_{x0}t \tag{4.5}$$

$$y = y_0 + v_{y0}t - \frac{1}{2}gt^2 \tag{4.6}$$

These equations form a set in which a large variety of problems can be solved. Further equations derived from these equations are merely "shortcuts." Many problems involved solving for the time elapsed during an event as an intermediate step. In these cases, it is useful to this ahead of time (if event duration is not of direct interest):

$$v_y^2 = v_{y0}^2 - 2g(\Delta y) \tag{4.7}$$

Of course for constant acceleration problems not related to gravity, one can replace g with $-a$.

4.4 Projectile Path Function

4.4.1 Surface-to-Surface Projectiles

Solving Eq. (4.5) for t, and substituting it into Eq. (4.6) yields

$$y(x) = tan(\theta_0)x - \frac{g}{2v_0^2cos^2(\theta_0)}x^2 \tag{4.8}$$

where

$$v_{x0} = v_0cos(\theta_0) \tag{4.9}$$

and

$$v_{y0} = v_0sin(\theta_0) \tag{4.10}$$

In addition, we have chosen the origin of the coordinate system to be located at the position where the projectile point of origination (uh, where the rock is thrown from).

By completing the squares, it follows that Eq. (4.8) may also be written

$$y(x) = \frac{v_0^2sin^2(\theta_0)}{2g} - \frac{g}{2v_0^2cos^2(\theta_0)}\left[x - \frac{v_0^2sin(\theta_0)cos(\theta_0)}{g}\right]^2 \tag{4.11}$$

This function represents a variable height function. By inspection, the maximum height is

$$y_{max} = \frac{v_0^2sin^2(\theta_0)}{2g} \tag{4.12}$$

By symmetry, the corresponding x value is half the range, so that

$$R = \frac{v_0^2sin(2\theta_0)}{g} \tag{4.13}$$

Again, solving Eq. (4.5) for t, and inserting the range formula Eq. (4.13) yields the flight time

$$T_{flight} = \frac{2v_0sin(\theta_0)}{g} \tag{4.14}$$

Example 4.1:

You hit a golf ball, and it leaves your club with a velocity of 30 m/s at an angle of 30 °.
a) How far does the ball go?
b) How high does the ball go?
c) How long does it take for the ball to get there?

Solution 4.1:

a) Using the range formula, Eq. (4.13), the answer is 79.5 m.
b) Using the formula for maximum projectile height, Eq. (4.12), the answer is 11.5 m.
c) Using Eq. (4.14), the flight time is 3.06 s. ♣

Example 4.2:

At what angle should you launch a surface-to-surface projectile to have it achieve maximum range?

Solution 4.2:

Examining the range formula, Eq. (4.13), it can be readily seen that the range is maximized when $sin(2\theta_0)$ *is the largest it can be. The largest value it can obtain is 1, and this happens when* $2\theta_0 = 90°$. *Thus, we should launch at* $45°$.

Alternatively, we can differentiate the range formula with respect to launch angle set it to zero, and solve for the corresponding launch angle. Proceeding in this manner,

$$\frac{\partial R}{\partial \theta_0} = \frac{v_0^2}{g} 2cos(2\theta_0) = 0 \tag{4.15}$$

which, again, is satisfied when $\theta_0 = 45°$. ♣

At times, it is of interest to rewrite Eq. (4.11) as

$$y(x) = y_{max} - \frac{4y_{max}}{R^2}\left[x - \frac{R}{2}\right]^2 \tag{4.16}$$

4.4.2 Non-Surface-to-Surface Projectiles

A generalized range formula can be obtained for projectiles that land at a different height than it was thrown. If a height, h, is defined to be positive when the landing height is greater than the initial height, then the generalized Range formula is

$$R = \frac{v_0^2 sin(2\theta_0)}{2g}\left[1 + \sqrt{1 - \frac{2gh}{v_0^2 sin^2(\theta_0)}}\right] \tag{4.17}$$

A general expression for the launch angle that achieves maximum range can be obtained when the ratio of the final potential energy to the initial x-component of the kinetic energy is small compared to unity. This expression is

$$\theta_{maxrange} = \frac{\pi}{4} + \frac{gh}{2v_0^2} \tag{4.18}$$

4.5 Conservation of Energy

Another result of interest in solving projectile motion problems is the conservation of energy. The conservation of energy relation can be obtained by using Eq. (4.7),

$$v_y^2 = v_{y0}^2 - 2g(\Delta y) \tag{4.19}$$

with

$$v_x^2 = v_{x0}^2 \tag{4.20}$$

Adding these equations yields

$$v^2 = v_0^2 - 2g(\Delta y) \tag{4.21}$$

which is the equation of interest. However, it is usually put into a slightly different form. Multiplying both sides by $\frac{1}{2}m$ results in

$$\boxed{\frac{1}{2}mv^2 + mgy = \frac{1}{2}mv_0^2 + mgy_0 = constant} \tag{4.22}$$

4.6 Temporary Notes

projectiles hit the ground at the same velocity that they are projected from :(What about baseball?

monkey in the tree

The horizontal and vertical components of projectile motion are independent. Therefore can drop a ball and shoot and gun at the same time and they will hit the ground at the same time.

61 degree launch gives same range at 29 degree launch

4.7 PROBLEMS

P4.1. Find the range formula for a projectile with a speed v_0 and an initial angle θ_0 thrown

1. up
2. down

a hill with slope $m = tan\phi$.