Contents

Tutorial 1

Differential Equations

Contents

1.1 Classifying Differential Equations

To classify differential equations we use a variety of terms, the most usual ones being,

1.1.1 Ordinary - Partial

A differential equation is called ordinary if the unknown function and its derivatives depend only on one independent variable. In a partial differential equation the unknown function and its derivatives depend on at least two independent variables.

Example: The time-dependent Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi = -i\hbar\frac{\partial \psi}{\partial t} \tag{1.1}$$

is a partial differential equation as it contains two independent variables x and t.[1]

1.1.2 Order

The order of a linear equation is determined by the order of the highest derivative in the equation.

Example: The equation describing the harmonic oscillator,

$$\frac{d^2x}{dt^2} + \omega_0^2 x(t) = 0 \tag{1.2}$$

[1]The partial derivative symbols may also have given it away!

is a second-order differential equation as the the highest derivative is of second order. Sometimes it is possible to solve a differential equation by reducing it to one of lesser order. However, this does not change the order of the initial equation.

One can always reduce a higher order ordinary differential equation to a set of first order equations. For example, we may define

$$y(t) \equiv \frac{dx}{dt} \tag{1.3}$$

This equation and the above equation may be rewritten

$$\frac{dv}{dt} = -\omega_0^2 x \tag{1.4}$$

$$\frac{dx}{dt} = v \tag{1.5}$$

1.1.3 Linear - Nonlinear

A linear differential equation contains only linear terms of the unknown function and its derivatives.

> **Definition: If y_1 and y_2 are solutions to a DE, the DE is linear if $c_1y_1 + c_2y_2$ is also a solution.**

Example: The differential equation describing a realistic pendulum,

$$\frac{d^2\phi}{dt^2} + \omega_0^2 sin(\phi(t)) = 0 \tag{1.6}$$

is a common example of a NONlinear DE.

1.1.4 Constant - Nonconstant Coefficients

1.1.5 Homogenous - Nonhomogenous

A differential equation is called homogenous if every term in it depends on the unknown function or its derivatives. It is inhomogenous if there is at least one term which depends only on the independent variables or is a constant different than zero.

Easy Check: Is zero a solution to the DE? If not, it is Nonhomogeneous.

Example: The differential equation of a driven harmonic oscillator,

$$\frac{d^2y}{dt^2} + \omega_0^2 y(t) = Acos(\omega t) \tag{1.7}$$

is inhomogenous as the term on the right only depends on t. If, in addition to being homogeneous, the differential equation is linear, then the general solution can be written in two parts: the "homogeneous" solution and the "particular" solution. We can trivially rewrite the above equation as

$$\frac{d^2(y_h + y_p)}{dt^2} + \omega_0^2 (y_h + y_p) = 0 + Acos(\omega t) \tag{1.8}$$

which can be rewritten

$$\underbrace{\left(\frac{d^2y_h}{dt^2} + \omega_0^2 y_h - 0\right)}_{=0} + \underbrace{\left(\frac{d^2y_p}{dt^2} + \omega_0^2 y_p - Acos(\omega t)\right)}_{=0} = 0 \tag{1.9}$$

It can readily be seen that the homogeneous solution has two integration constants. Thus, the particular solution will not have an integration constants.

1.2 One You're Expected to Know

$$\frac{d^2y}{dt^2} + \omega_0^2 y(t) = 0 \tag{1.10}$$

The general solution is

$$y(t) = Acos(\omega_0 t) + Bsin(\omega_0 t) \tag{1.11}$$

for an arbitrary constants A and B.[2]

1.3 Physical Systems

no solutions in, no solutions out

1.3.1 Three types of solutions

analytical solutions, series solutions, computer solutions

1. Classify the following

$$\frac{d^2\phi}{dt^2} + \omega_0^2 sin(\phi(t)) = 0 \tag{1.12}$$

2. Classify the following

$$\frac{d^2y}{dt^2} + \omega_0^2 y(t) = 0 \tag{1.13}$$

3. Classify the following

$$\frac{d^2y}{dt^2} + \omega_0^2 y(t) = Acos(\omega t) \tag{1.14}$$

4. Classify the following

$$\left(\frac{d^2y}{dt^2}\right)^2 + \omega_0^2 y(t) = Acos(\omega t) \tag{1.15}$$

[2] Yes, A can be a complex constant

5. Consider the following first order LODE:

$$\frac{dy}{dt} + 3ty(t) = 0 \tag{1.16}$$

Get all of the y terms on one side of the equation and get all of the t terms on the other side of the equation. Integrate both sides and solve for $y(t)$.

6. Find the general solution to the following $2nd$ order LODE:

$$\frac{d^2x}{dt^2} + 3x(t) = 0 \tag{1.17}$$

7. Rewrite the following second order ODE as two first order ODEs:

$$\frac{d^2\phi}{dt^2} + \omega_0^2 sin(\phi(t)) = 0 \tag{1.18}$$