PHYS 202 Notes on circular and rotational motion

Crowell is rather terse in his treatment of circular motion. I would like to fill in some of the gaps.

1 Circular motion, torque, angular momentum

Some definitions

The diagram at right represents a portion of a circle (part of a “pie”) with radius $r$, arc length $s$, and angle $\theta$. If we double $r$, keeping $\theta$ the same, the arc length doubles as well. If we double the angle, keeping $r$ the same, again the arc length doubles. So $s$ is proportional to both $r$ and $\theta$:

$$s \propto r \theta \quad \text{or} \quad s = (\text{constant})r\theta$$

simplest thing to do is just let the constant be 1. Then

$$s = r\theta$$  \hspace{1cm} (1)

This means that $\theta$ is the ratio $s/r$, which is defined as the angle in radians. The radian is a “natural” unit for angle, much more so than degrees.

Now, what corresponds to a complete, or $360^\circ$, rotation? In that case, $s$ is the circumference $c$ of the circle, and we know that this is $c = 2\pi r$. The angle then is

$$\theta = \frac{s}{r} = \frac{c}{r} = \frac{2\pi r}{r} = 2\pi \text{ radians}$$

Therefore an angle of $360^\circ$ corresponds to $2\pi$ radians. The usual conversion factor is $\pi$ radians / $180^\circ$ or its inverse, depending on which way you are converting.

Now let us consider the circular motion of a point mass. Suppose a mass is constrained to move in a circle, such as in the diagram at right. We could define the angular position as the angle $\theta$ with respect to some axis, like the $x$-axis. (That’s what we did in the diagram.) The rate of change of this angle is defined as the angular velocity $\omega$:

$$\omega = \frac{\Delta \theta}{\Delta t}$$

This is the same way we have seen velocity defined before:

$$v = \frac{\Delta s}{\Delta t}$$

where in this case $\Delta s$ is the distance traveled along the arc, in some time $\Delta t$. 
To relate these, we may write

\[
\Delta s = r \Delta \theta \quad \text{and divide by } \Delta t : \\
\frac{\Delta s}{\Delta t} = r \frac{\Delta \theta}{\Delta t} \\
v = r \omega
\] (2)

We can go one better now: suppose the mass is accelerating with some rate \(a\), going faster and faster along the circle. This is \textit{not} centripetal acceleration, which is toward the center. It is doing that too, even when the speed is constant. But we are interested now in the “tangential” acceleration \(a\). The angular velocity \(\omega\) is also getting faster. We may take Equation 2 and divide again by \(\Delta t\):

\[
\Delta v = r \frac{\Delta \omega}{\Delta t} \\
\frac{\Delta v}{\Delta t} = r \frac{\Delta \omega}{\Delta t} \\
a = r \alpha
\] (3)

where \(\alpha\) (alpha) is defined as the rate of change of angular velocity and is called the \textit{angular acceleration}.

We now have three equations relating “linear” position on the arc to angular position:

\[
s = r \theta \\
v = r \omega \\
a = r \alpha
\]

1.1 Torque

\textit{Torque} is the twisting type of “force” that you apply when you want to make something rotate; for example removing a lid from a jar. We may define it as a force applied at some distance \(r\) from a center of rotation. To use a wrench to tighten a nut, you apply a force near the end of the handle, and at a right angle to the handle:

We note some points with which you hopefully agree: (1) To increase the torque (i.e. tighten the nut more) we could apply more force. (2) However, we could also just keep the force the
same and increase the radius at which we apply the force. When using a wrench, we can do this with a “cheater bar”, a metal tube that you slip over the handle to increase the effective length of the handle. (This is also a good way to break the wrench.) Thus, the torque seems to depend on both the force and the radius. We define it as the product of these:

\[ \tau = F_\perp r \]  

(4)

where \( F_\perp \) is the force at a right angle to the radius. If the force vector points some other direction, then \( F_\perp \) is the component of that force which is perpendicular to the radius vector \( \vec{r} \). As we will see later, this component is \( F_\perp = F \sin \theta \) where \( \theta \) is the angle between the moment arm and the force vector. This gives the general expression

\[ \tau = Fr \sin \theta \]  

(5)

Conditions for equilibrium We already know that if a system is in equilibrium (that is, not accelerating; usually we want it sitting still) then the sum of forces on it must be zero. We call this the **first condition for equilibrium**: 

\[ \sum \vec{F}_x = 0 \quad \text{and} \quad \sum \vec{F}_y = 0 \]

That is, the sum of components of forces must be zero both in the \( x \) and \( y \)-directions. But suppose that we have an extended body (not a point mass) with two equal and opposite forces on it, as shown:

The first condition for equilibrium will be satisfied, but clearly this object is going to rotate: it is not at equilibrium. We have to introduce another condition, that the sum of torques is also zero.

\[ \sum \vec{\tau} = 0 \]

This is the **second condition for equilibrium**. By the way, a pair of equal and opposite forces like that shown, which produce a torque, is called a **couple**.

1.1.1 Center of mass

Suppose we have three masses on a light rod, at positions \( x_1 \), \( x_2 \), and \( x_3 \). The rod is horizontal and is pivoting about a point near the left end, as shown.
Each mass exerts a torque about the pivot, the total torque being

\[ \tau = m_1g x_1 + m_2g x_2 + m_3g x_3 \]

We wish to replace all three masses with a mass equal to their sum, and positioned so that the torque about the pivot point is the same. Let’s call the position \( X_{CM} \). Then

\[
(m_1 + m_2 + m_3)g X_{CM} = m_1g x_1 + m_2g x_2 + m_3g x_3 \\
(m_1 + m_2 + m_3)X_{CM} = m_1 x_1 + m_2 x_2 + m_3 x_3
\]

Equation 6 defines the center of mass, \( X_{CM} \). For any collection of masses, or for any extended body, we can often replace the body with a point mass located at the center of mass and with mass equal to the total of the original system.

1.1.2 Example

Suppose we have a meterstick, and poke a needle through a hole at the \( x = 30 \text{ cm} \) mark, so that the meter stick can pivot about this point. We then hang a mass \( m = 100 \text{ grams} \) somewhere between the 0 and 30 cm marks, to balance the meterstick. We find that it balances then \( m \) is at \( x = 5.0 \text{ cm} \). We can use this information to find the mass \( M \) of the meterstick.

To analyse this situation, we draw a force diagram, as above. We can replace the mass of the meterstick with a point mass located at the center of mass. For a uniform object like the stick, the center of mass is simply in the middle. We show this in the force diagram. The counterclockwise and clockwise torques must be equal and opposite, in order that the system is balanced. We write:

\[
\tau_{CW} = \tau_{CCW} \\
M g(20 \text{ cm}) = mg(25 \text{ cm})
\]

\[
M = \frac{25}{20} m = 1.25(100 \text{ grams}) \\
M = 125 \text{ grams}
\]

Note that we did not have to worry a lot about using only SI units, since in this kind of calculation the distance units cancel. Note also that the acceleration of gravity canceled out, simplifying the arithmetic.
We might also ask: what is the upward force for \( F \) on the pivot. By the first condition for equilibrium, the total upward force must be equal and opposite to the total downward force.

\[
F_{up} = F_{down} \\
F = mg + Mg = 0.98 \text{ N} + 1.23 \text{ N} \\
= 2.21 \text{ N}
\]

We could have found the upward force \( F \) by using torque and the second condition for equilibrium. Just for illustration, let’s show one of the many possible ways to do that. Let us put a pivot point, for the purpose of calculation, at 5.0 cm, the location of \( m \). One reason for doing this is that it eliminates \( m \) from the calculation, since the moment arm (i.e. radius) for \( mg \) is zero. This simplifies the calculation. We now equate the CW and CCW torques:

\[
\tau_{CW} = \tau_{CCW} \\
Mg(45 \text{ cm}) = F(25 \text{ mg}) \\
F = \frac{45}{25}(1.23 \text{ N}) \\
F = 2.21 \text{ N}
\]

as before. It is important to realize that the choice of the pivot point is arbitrary. No matter what point you choose, the total torque about this point must be zero. Therefore, we choose a point that eliminates the torque(s) from as many forces as possible.

### 1.2 Torque calculations, revisited

We have already defined torque as \( \tau = F_r \), where \( F_\perp \) is the component of the force perpendicular to the moment arm (that is, the line going from the pivot point to the point the force is applied.) This illustrated below. The left-hand diagram makes it clear that \( F_\perp = F \sin \theta \), leading to the expression

\[
\tau = Fr \sin \theta
\]

which is Equation 5.

However, there is another way to find the torque, which is also illustrated. Instead of finding the perpendicular component of the force, one extends a line along the direction of the force vector, and draws a line from the pivot point to the “force vector” line, and perpendicular to the latter. We call this second line \( r_\perp \) since it is a moment arm which is perpendicular to the force. Looking at the diagram, we see that

\[
r_\perp = r \sin \theta
\]

But now look at this equation for torque, just before this. Substituting, we can rewrite that equation as

\[
\tau = Fr_\perp
\]
Both ways are used. Often, one way is considerably easier than the other. For example, suppose we examine a 10-kg door hanging on its hinges. We want to find the torque, using one of the hinges as a pivot point. This is illustrated below. At left, we give some dimensions, and at right we draw a separate force diagram, because a single diagram would be too cluttered. “CM” is the center of mass, in the middle of the door.

Let’s try to use the expression $\tau = F_\perp r$ to find the torque due to the weight, applied at the center of mass. First, we have to find the distance from the pivot point (bottom hinge) to the center of mass. This can be done with the Pythagorean relation. Then, you have to find the component of the weight vector, $m\vec{g}$, that is perpendicular to the line from the hinge to center of mass. If you don’t think this is complex, then stop right now and try to work it out.

Instead, let’s try the relation $\tau = Fr_\perp$. The force $F$ is just $mg$. The line from the pivot point to the $m\vec{g}$ vector is simply a horizontal line half the width of the door. Since $M = 10$ kg, the torque works out to

$$\tau = (10 \text{ kg})(9.80 \text{ m/s}^2)(0.75 \text{ m}) = 74 \text{ N-m}$$

You decide which is easier.

To learn some more from this example, let’s use the second condition for equilibrium to find the horizontal force on the top hinge. This is labeled $F_1$ in the diagram. The forces that go through the bottom hinge can contribute nothing to the torque. The vertical force on the top hinge exerts no torque, either. The only two forces contributing to torque about
the bottom hinge are $F_1$ and $mg$. Then

\[
\tau_{\text{CCW}} = \tau_{\text{CW}} \\
F_1(1.5 \text{ m}) = mg(0.75 \text{ m}) = 73.5 \text{ N-m} \\
F_1 = 49 \text{ N}
\]

Now, what is the horizontal force on the bottom hinge, $F_2$? By the first condition for equilibrium, the sum of horizontal forces must add up to zero. But the only horizontal forces are $F_1$ and $F_2$. It is clear that these must be equal and opposite, so $F_2$ is also 49 N, but applied to the right.

What about the vertical components, $F_3$ and $F_4$? By the first condition for equilibrium, these must add up to the weight, $mg$. Ideally, each hinge would support half the weight. In practice, however, it is pretty difficult to make this happen. Usually one of the hinges will support more of the weight than the other. This makes no difference to our calculation of $F_1$ and $F_2$, however. With only the information given, there is no unique solution for splitting up the vertical forces applied by the two hinges.

### 1.3 Angular momentum

Suppose again we have a mass constrained to move in a circle, like a bead on a circular wire loop. Let us apply a force $F$ to the mass in a direction tangent to the arc, as in the diagram. (As the bead moves, we have to adjust the direction of the force so it always is “grazing” the circle.) We know from Newton’s second law that

\[ F = ma \]

We want to relate this somehow to the angular motion. First, we know that $a = r\alpha$, so

\[ F = m r \alpha \]

Multiplying both sides by $r$ gives

\[ Fr = (mr^2)\alpha \]

But $Fr$ is the torque, so we obtain

\[ \tau = (mr^2)\alpha \]

We could call this the “angular form” of Newton’s second law. It looks like $F = ma$, except we have torque instead of force, $\alpha$ instead of $a$, and the strange quantity $mr^2$ instead of mass.

So far, there is no advantage to using this “angular form” of the second law instead of the usual form. However, when we start dealing with extended bodies — that is, masses that are not just point masses — it turns out to be useful. Suppose we have three point masses hooked together with “massless” rods, to form a rigid structure. We wish to apply a torque to make this structure rotate about some point.
In the diagram, we have a force $F$ applied at some radius $r$. It is applied to some point on the structure, not to a particular mass. To figure out the rate of acceleration of this structure, about the pivot point, we have to find a TOTAL value for the quantity $mr^2$. That is, we have to add up the $mr^2$ values for all three masses, and then we can apply the angular form of the Second Law:

$$rF = \tau = (m_1R_1^2 + m_2R_2^2 + m_3R_3^2)\alpha$$

We give the quantity in parentheses a name: the rotational inertia of the system. Its symbol is $I$. The angular form of the Second Law becomes:

$$\tau = I\alpha \tag{8}$$

Now, most systems we work with are not three masses held together with rods: they are solid bodies of various shapes. But to find the rotational inertia of such systems, we still have to “add up” all the $mr^2$ values of the atoms making up the mass. If you did it one atom at a time, this would be a big job. There are ways to make the task easier, and for many common shapes, the job is done for you. You can look up the expression for $I$, for many common shapes, as we will see presently.

Thus far, we have not said anything about “angular momentum.” First, consider something we already know about: linear momentum, $p = mv$. Properly written, Newton’s second law states:

$$F_{net} = \frac{\Delta p}{\Delta t} \tag{9}$$

When mass is constant, we can write

$$F_{net} = \frac{\Delta (mv)}{\Delta t} = m\frac{\Delta (v)}{\Delta t} = ma$$

But that is a special case.

Our expression for the angular form of Newton’s second law may be written as follows:

$$\tau = I\alpha = I \frac{\Delta \omega}{\Delta t}$$

$$\tau = I \frac{\Delta (I\omega)}{\Delta t}$$

Compare this with Equation 9. In place of momentum, we have the quantity $I\omega$. The time rate of change of this quantity gives us the torque. It makes sense, then, that $I\omega$ is called the angular momentum, $L$:

$$L = I\omega \quad \text{and} \quad \tau = \frac{\Delta L}{\Delta t} \tag{10}$$
Look again at Equation 9. It implies that if there are no external forces on a system ($F_{\text{net}} = 0$), then the total momentum of that system cannot change. We can use this to analyze collisions, for example. Compare this with Equation 11. It implies that if the net torque is zero, the angular momentum of the system cannot change. We see the effects of this a lot more often than for the linear case, because it is very easy to change the rotational inertia, $I$. Suppose a system has some angular momentum and the inertia changes from $I_1$ to a value $I_2$. If the angular momentum is conserved, then

$$I_1 \omega_1 = I_2 \omega_2$$

For example, when an ice skater is spinning with one leg and arm extended, her rotational inertia $I_1$ is large. When she brings her leg and arms close to her body, the value of $I$ decreases (since the average of $mr^2$ decreases.) If this happens, $\omega$ must get large, since $I\omega$ stays the same. Therefore she spins a lot faster. One sees this effect dramatically in the fouetté (or fouetté en tournant) of classical ballet. (In the famous black swan variation of Swan Lake, the ballerina, if she has the strength, does 32 consecutive fouettés!)

**Rotational inertia for various shapes**

For a hoop rotating about its center, all pieces of mass are at the same radius $R$. Therefore if you sum up all values of $\Delta m r^2$, you just get the total mass times $R^2$: $I_{\text{hoop}} = MR^2$. For a disk, however, much of the mass is at smaller radii, which decreases the $I$-value, compared to a hoop of the same mass and radius. We won’t do the math, but it turns out that for a disk or cylinder rotating about its center, $I_{\text{disk}} = (1/2)MR^2$. We here present a table summarizing $I$ for a few common shapes. In each case, you must be careful to specify the axis about which rotation is happening! That is why we present diagrams for each.

<table>
<thead>
<tr>
<th>Object</th>
<th>Rotational axis</th>
<th>Diagram</th>
<th>Rotational Inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hoop</td>
<td>Through center, perp. to plane</td>
<td><img src="image" alt="Diagram" /></td>
<td>$MR^2$</td>
</tr>
<tr>
<td>Cylinder or disk</td>
<td>Through axis of symmetry</td>
<td><img src="image" alt="Diagram" /></td>
<td>$\frac{1}{2}MR^2$</td>
</tr>
<tr>
<td>Rod of length $L$</td>
<td>About end</td>
<td><img src="image" alt="Diagram" /></td>
<td>$\frac{1}{3}ML^2$</td>
</tr>
<tr>
<td>Rod of length $L$</td>
<td>Through center</td>
<td><img src="image" alt="Diagram" /></td>
<td>$\frac{1}{12}ML^2$</td>
</tr>
<tr>
<td>Solid sphere</td>
<td>Through center</td>
<td><img src="image" alt="Diagram" /></td>
<td>$\frac{2}{5}MR^2$</td>
</tr>
<tr>
<td>Spherical shell</td>
<td>Through center</td>
<td><img src="image" alt="Diagram" /></td>
<td>$\frac{2}{3}MR^2$</td>
</tr>
<tr>
<td>Square plate</td>
<td>Through center, perp. to plane</td>
<td><img src="image" alt="Diagram" /></td>
<td>$\frac{1}{6}ML^2$</td>
</tr>
</tbody>
</table>
1.3.1 The special case of a point mass

For a point mass moving in a circle, the inertia is $mr^2$, so we may write the angular momentum as

$$L = I\omega = mr^2 \left(\frac{v}{r}\right) = mvr$$

For cases involving orbiting bodies, like satellites around the earth, this is sometimes simpler.

1.4 Analogies between linear and circular motion

Let us again consider a point mass moving in a circle of radius $r$, and find its kinetic energy. We start with the kinetic energy in terms of speed $v$, and want to change this expression to an “angular” form.

$$KE = \frac{1}{2}mv^2$$
$$= \frac{1}{2}m(r\omega)^2 = \frac{1}{2}mr^2\omega^2$$
$$= \frac{1}{2}I\omega^2 \quad \text{since} \quad I = mr^2 \text{ for a point mass.}$$

This can be generalized to rotating systems of extended masses, as well. We note that the angular expression is just the same, except for replacing $m$ with $I$, and $v$ with $\omega$. This kind of analogy applies to a great many of the quantities involved with linear and angular motion.

Let’s make a table of such analogies.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Linear form</th>
<th>Angular form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Displacement</td>
<td>$x$ or $s$</td>
<td>$\theta$</td>
</tr>
<tr>
<td>Velocity</td>
<td>$v$</td>
<td>$\omega$</td>
</tr>
<tr>
<td>Acceleration</td>
<td>$a$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>Force</td>
<td>$F$</td>
<td>$\tau = F_\perp r$ (torque)</td>
</tr>
<tr>
<td>Momentum</td>
<td>$p = mv$</td>
<td>$L = I\omega$</td>
</tr>
<tr>
<td>Kinetic Energy</td>
<td>$KE = \frac{1}{2}mv^2$</td>
<td>$KE = \frac{1}{2}I\omega^2$</td>
</tr>
<tr>
<td>Work</td>
<td>$W = Fx$</td>
<td>$W = \tau\theta$</td>
</tr>
<tr>
<td>Equation of motion</td>
<td>$v_f = v_o + at$</td>
<td>$\omega_f = \omega_o + \alpha t$</td>
</tr>
<tr>
<td>Equation of motion</td>
<td>$x - x_o = v_o t + \frac{1}{2}at^2$</td>
<td>$\theta - \theta_o = \omega_o t + \frac{1}{2}\alpha t^2$</td>
</tr>
<tr>
<td>Equation of motion</td>
<td>$v_f^2 - v_o^2 = 2a(x - x_o)$</td>
<td>$\omega_f^2 - \omega_o^2 = 2\alpha(\theta - \theta_o)$</td>
</tr>
</tbody>
</table>

Doubtless you may think of some others to go with these.
1.5 More with kinetic energy

Consider a grinding wheel sitting on a bench and spinning on its axis. It has kinetic energy $(1/2)I\omega^2$. Suppose you throw the wheel with some velocity $v$, without spinning it. It has kinetic energy again, but this time it is due to translation rather than rotation: $(1/2)mv^2$. Finally, suppose the wheel is rolling along the floor without slipping. This time, it has both kinds of kinetic energy: translational and rotational. Then

$$KE = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

The kinetic energy is “split” between the two types. The nature of the division of energy depends on the shape of the object. Let us give an example.

Suppose a hoop rolls across the floor, as in the diagram. Let’s find an expression for the total kinetic energy as a function ONLY of the velocity. The hoop is not slipping and is rolling on its “outside” radius, so $v = r\omega$. The rotational inertia of a hoop is $I = Mr^2$. Making substitutions,

$$KE = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}(mr^2)\left(\frac{v}{r}\right)^2$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}mv^2$$

$$= mv^2$$

We see that the rotational and translational parts are the same, both equal to $(1/2)mv^2$, giving a total kinetic energy of $mv^2$.

Now, suppose we repeated this, but for a rolling disk instead. The only difference would be the expression for inertia: $I = (1/2)mr^2$. You may verify that this time, we would obtain

$$KE (disk) = \frac{3}{4}mv^2$$

Of course, we could have found the energy in terms of $I$ and $\omega$, but for a rolling object we would generally measure its linear velocity, not angular velocity.
1.6 The Parallel Axis Theorem

This is not covered by many textbooks, but it comes in handy. It is of use in the Physical Pendulum lab, for instance. Consider the moon in its path around the earth. To find its kinetic energy, say, we can treat it as a point mass to a pretty good approximation. However, note that the moon is rotating on its axis, once every revolution. (That is why the same side of the moon faces the earth all the time.) Wouldn’t this rotation produced a rotational kinetic energy, which would add to the translational kinetic energy that we usually think of? The answer is yes, although in the case of the moon, the rotation is so slow that it makes little difference.

Instead, think of a hoop rotating about one edge. We can see that the center of mass (in the middle) is revolving about the pivot point, but at the same time, the hoop is changing its orientation, or rotating. We might guess that the overall angular momentum would be that due to the motion of the center of mass, as a point particle, plus that due to the rotation of the object about its center of mass. This turns out to be correct, and can be rigorously shown. We may write the rotational inertia as the sum of the two corresponding types:

\[ I_{\text{total}} = I_{CM} + Mr_{CM}^2 \]  

where \( I_{CM} \) is the rotational inertia measured about the center of mass, \( M \) is the total mass, and \( r_{CM} \) is the distance from the pivot point to the center of mass.

In our hoop example, we know that \( I_{CM} = MR^2 \) If we rotate it about the edge, then, \( r_{CM} = R \), so

\[ I = MR^2 + MR^2 = 2MR^2 \]

The rotational inertia for a hoop pivoted about its outside is twice that for a hoop rotating about its center. For a disk, this is

\[ I = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2 \]

Here is another example. A square plate rotating about its center of mass has a rotational inertia \( I = (1/6)ML^2 \). What is the inertia for a square plate pivoted about one corner? Well, the distance from a corner to the center is \( L/\sqrt{2} \), so

\[ I = \frac{1}{6}ML^2 + M \left( \frac{L}{\sqrt{2}} \right)^2 = \frac{2}{3}ML^2 \]

One final example. In a homework problem, we used a rod, pivoting about a point 1/3 of the way from one end. Suppose the rod has length \( \ell \) and total mass \( M \). We could think of two ways to find the moment of inertia.
(1) “Break” the rod into two parts, one of length $\ell/3$ and another of length $2\ell/3$, with corresponding masses. The inertia of a rod pivoted about its end is $(1/3)M\ell^2$. Thus, the sum of the inertias of the two rods is

$$I = \frac{1}{3} \left( \frac{M}{3} \right) \left( \frac{\ell}{3} \right)^2 + \frac{1}{3} \left( \frac{2M}{3} \right) \left( \frac{2\ell}{3} \right)^2 = \frac{1}{9}M\ell^2$$

(2) Use the parallel axis theorem. The distance from the pivot to the center of mass is $\ell/6$. The rotational inertia of a rod pivoted around its center is $(1/12)M\ell^2$. So

$$I = \frac{1}{12}M\ell^2 + M \left( \frac{\ell}{6} \right)^2 = \frac{1}{9}M\ell^2$$

as before.

## 2 Vibrations and waves – relating simple harmonic motion to circular motion

### 2.1 Mass and spring

The archetype of oscillating systems is a mass on a spring. For a system to oscillate, two things are necessary: a mass or other source of inertia, and a restoring force. In the diagram below, the mass slides on a frictionless surface, so no energy is lost.

![Diagram of mass on spring](image)

We know from previous studies that an ideal spring obeys the relation

$$F = -k(x - x_o)$$  \hspace{1cm} (13)

where $F$ is the force exerted BY the spring in response to a displacement $x - x_o$. ($x_o$ is the relaxed position of the spring.) That is, the spring’s force is in a direction opposite to the displacement direction. We say that it provides a restoring force, trying to “restore” the
mass (or whatever) to its equilibrium position. The equation above is called Hooke’s Law, after Robert Hooke, a contemporary of Isaac Newton.

If you pull the mass to one side and let go, it oscillates. The spring pulls the mass to the equilibrium position, but because of the mass’s inertia, it “overshoots” and ends up displaced in the other direction. The spring reverses its force, stops, the mass, and accelerates it back to the equilibrium position. The mass overshoots, and the cycle repeats. An object oscillating in this way, when the spring obeys Hooke’s Law, is said to exhibit simple harmonic motion, often abbreviated SHO.

2.2 Relation to circular motion

Let’s consider a system involving a mass sliding on a frictionless surface as before, but now suppose that a turntable of some kind is placed over the mass, with its center above the equilibrium position. The sun is directly overhead, and an object on the turntable casts a shadow on the mass below. If we adjust the radius of the circle and the angular velocity of the object, we can get the shadow to exactly follow the motion of the oscillating mass. Therefore, we can describe the oscillation in terms of the radius of the turntable and its angular velocity $\omega$. The linear motion of something undergoing simple harmonic motion is the projection, along one axis, of an object moving in a circle.

Now let’s make this more quantitative.

2.3 Equations for simple harmonic motion

Take a look at the diagram on the next page. The circle has a radius $A$, so the mass below can oscillate between positions $-A$ and $+A$: we call $A$ the amplitude.
For a particular angle $\theta$ (measured with reference to the $x$-axis in our diagram), we can find the position $x$ of the oscillating mass using the right triangle. The hypotenuse is $A$ and the displacement is $x$, so $x/A$ is the cosine of $\theta$. Thus,

$$x = A \cos \theta$$

Now, suppose that $\theta = 0$ when time $t$ is also zero. Then by the definition of angular velocity, $\theta = \omega t$. Therefore we can write

$$x = A \cos (\omega t) \quad (14)$$

At this point we should note that the vertical projection of the dot on the circle would also exhibit simple harmonic motion. The vertical displacement $y$ in the diagram is equal to $A \sin \theta$, so we may also describe simple harmonic motion as

$$x = A \sin (\omega t) \quad (15)$$

Both expressions are used. Some textbooks prefer one, and some the other. I personally prefer to use the sine function since that way, $x = 0$ at time $t = 0$. This seems simpler and more elegant to me than using the cosine. But an engineer friend of mine says that no, if you pull the mass to one side and let go at $t = 0$, $x$ must have some nonzero value at $t = 0$. So he prefers the cosine expression. It’s a religious argument.

### 2.4 Maximum velocity for SHO

The oscillating mass accelerates toward the equilibrium point, except that $a = 0$ right at that point. Therefore when the mass is moving toward the point, it speeds up until it reaches the equilibrium point. Then the maximum speed of the mass will occur at the equilibrium point. To find this speed, we look at our diagram: the speed of the mass at the center of travel is equal to the speed of the dot on the circle, either at the top or bottom of the circle. That is because the dot is moving horizontally at that point: it has no vertical component of velocity. This velocity is equal to the tangential velocity of the dot on the circle (i.e. its speed) $v = r \omega$. Thus the maximum speed is

$$v_{\text{max}} = A \omega \quad (16)$$

### 2.5 Maximum acceleration for SHO

The maximum acceleration occurs at the ends of travel, because the force of the spring is greatest there. Looking at the circle, we see that this is when the dot crosses the $x$-axis. From what we know about centripetal acceleration, we see that the dot is accelerating horizontally,
toward the center of the circle. This is the same direction as the acceleration of the oscillating mass. This acceleration is simple \( a_c = r\omega^2 \), so the maximum acceleration of the mass is

\[
a_{\text{max}} = A\omega^2
\]  

Here, \( a_{\text{max}} \) is a magnitude; it is a positive quantity. In a system, the sign of the actual acceleration is opposite sign of the displacement. (See Equations 24 and 26, and the graphs which follow them.)

### 2.6 Energy in a SHO system

A spring’s potential energy is

\[
PE = \frac{1}{2}kx^2
\]

as we learned last term. (One can readily show this by finding the area under the curve of the spring’s force vs. displacement graph.) A mass oscillating on a spring has two kinds of energy, KE and PE, and its total energy is the sum of these:

\[
E = KE + PE = \frac{1}{2}mv^2 + \frac{1}{2}kx^2
\]

At the center, PE = 0, so all energy is kinetic. At the extremes of motion \( (x = \pm A) \), KE = 0, so all energy is potential. The total energy does not change if there is no friction. It is clear, then, that

\[
E = \frac{1}{2}kA^2 \quad \text{at the extremes, and}
\]

\[
E = \frac{1}{2}mv_{\text{max}}^2 \quad \text{in the middle (equilibrium position)}
\]  

### 2.7 Finding the angular velocity of a mass/spring system

Let us exploit this knowledge to find yet another important relation. Look at Hooke’s law (Equation 13): \( F = -kx \). If the force on the oscillating mass is only due to the spring, we may use Newton’s second law to write

\[
F_{\text{net}} = -kx = ma
\]

Thus,

\[
a = -\frac{k}{m}x
\]

At the ends of travel, say when \( x = \pm A \), the acceleration is maximum:

\[
a_{\text{max}} = -\frac{k}{m}A
\]

But now let us go back and look at Equation 17: \( a_{\text{max}} = -A\omega^2 \). Comparing these two equations, we see that

\[
\frac{k}{m} = \omega^2
\]
Therefore, the angular velocity is

\[ \omega = \sqrt{\frac{k}{m}} \]  

Equations 20 and 21 are worth remembering. We will use them a lot this term. The former, which relates acceleration and displacement, is often called the “equation of motion” for a system.

2.8 Java applets which illustrates these concepts

This might be a good time to use a computer and view some animated applets which are good illustrations of this discussion. I suggest the following URLs:

- http://www.phy.ntnu.edu.tw/ntnujava/index.php?topic=148 This one also shows a graph of position versus time, generating a sine-wave graph.
- http://surendranath.tripod.com/Applets/Oscillations/SHM/SHMApplet.html This one has better color and is simpler, but does not make a graph.
- http://qbxB6.ltu.edu/s_schneider/physlets/main/mass_spr_vfric.shtml It does not have a circle, but this is a fun applet since you can use the mouse to position the mass on the spring, and you can add two types of friction to see what happens.

2.9 The simple pendulum

A simple pendulum is a mass, considered as a point mass, hung on the end of a string. The mass is often called a “bob.” In the figure, the pendulum string of length \( \ell \) is at an angle \( \theta \) from the vertical \(^1\). The pendulum bob is accelerating at a right angle to the string’s line. Looking at the free-body diagram, we see that the component of the weight, \( mg \), in this direction is just \( mg \sin \theta \). Therefore the torque on the pendulum is

\[ \tau = LF_\perp = Lmg \sin \theta \]

Putting this into the angular form of Newton’s second law gives us

\[ \tau = I \alpha \]
\[ rF_\perp = I \alpha \]
\[ \ell mg \sin \theta = (m \ell^2) \alpha \]
\[ g \sin \theta = L \alpha \]

\(^1\)We don’t use \( L \), since that symbol is reserved for angular momentum in this material.
Now, let us restrict the motion to **small angles**, less than 10°. This is important, because it allows us to use the small-angle approximation for the sine function. For small angles, \( \sin \theta \approx \theta \). (Of course, \( \theta \) must be in radians.) Try this using your calculator. For angles less than 0.2 radians, the approximation is not bad. At \( \theta = 0.2 \) rad, \( \sin \theta \) is less than one percent different from \( \theta \). For smaller angles, the approximation is better yet.

Using this approximation in our equation above, we can obtain

\[
g\theta = \ell \alpha
\]

\[
\alpha = \frac{g}{\ell} \theta
\]

Now, compare Equations 20 and 22. They have the same form. The only difference is that the linear terms \( x \) and \( a \) are replaced with their angular counterparts, \( \theta \) and \( \alpha \). **But both equations state that acceleration is proportional to displacement.** In the case of Equation 20, we found that the angular velocity could be found from the constant of proportionality, \( k/m \). This is Equation 21: \( \omega = \sqrt{k/m} \). Looking at our “angular” equation, we reason that the angular velocity (or angular frequency) \( \omega \) could be found the same way, using the constant of proportionality:

\[
\omega = \sqrt{\frac{g}{\ell}} \text{ Angular frequency of a simple pendulum (22)}
\]

### 2.9.1 Digression – another derivation

We could have done this another way. For small angles, the pendulum bob’s path does not vary much from a straight horizontal line. The horizontal displacement is just \( x = \ell \sin \theta \).

Now, what is the restoring force? For small angles, the tension in the string does not change much from the weight: \( T \approx mg \). Drawing a diagram, you should be able to convince yourself that the horizontal component of the tension is

\[
F_x = T \sin \theta
\]

Since \( \sin \theta = x/\ell \) and \( T \approx mg \), we can write

\[
F_x = mg \frac{x}{\ell}
\]

Now we apply Newton’s second law:

\[
F = ma
\]

\[
mg \frac{x}{\ell} = ma
\]

\[
a = \frac{g}{\ell} x
\]

Again, we have an equation in which acceleration is proportional to displacement. This looks just like Equation 20 except that \( k/m \) is replaced by \( g/\ell \). We end up with the same expression for angular velocity: \( \omega = \sqrt{g/\ell} \).
2.9.2 Keep this straight!

With a pendulum, there are two kinds of angular velocity.

1. The one we have just been dealing with. This is the constant angular velocity of a dot on an imaginary circle, the kind you viewed in the Java applets. This is the $\omega$ which we calculate as $\sqrt{k/m}$. We use this to find the frequency, $f = \omega/(2\pi)$.

2. The angular velocity of the string holding up the pendulum bob. This angular velocity varies, oscillating back and forth between two values. We sometimes need to know it, such as cases where we calculate the kinetic energy of the pendulum.

2.10 The Physical Pendulum

The term “physical pendulum” can describe any pendulum in which an extended body is swinging back and force on a pivot. Some textbooks restrict it to a rod pivoting about its end, but the term is more general than that. For illustration, we will examine the case of a rod.

This rod pivots about its end. There are forces at the pivot, $F_x$ and $F_y$, but they cannot exert a torque about the pivot, so they do not enter into the analysis. The torque is just

$$\tau = \frac{\ell}{2} mg \sin \theta$$

We again use the small-angle approximation for the sine function, so that

$$\tau \approx \frac{\ell}{2} mg \theta$$

We now need an “equation of motion,” which we obtain by writing Newton’s second law, angular form:

$$\tau = I \alpha$$

$$\frac{\ell}{2} mg \theta = \frac{1}{3} mL^2 \alpha$$

$$\alpha = \frac{3g \theta}{2\ell}$$

Using the analogy with Equation 20, as we did before, we find the angular frequency to be

$$\omega = \sqrt{\frac{3g}{2\ell}}$$

(23)

The frequency, as usual, would be $\omega/2\pi$, or

$$f = \frac{1}{2\pi} \sqrt{\frac{3g}{2\ell}}$$
This, again, is not the only physical pendulum possible. You might imagine lots of other configurations. For example, a hoop hanging on a peg and swinging back and forth is a physical pendulum. A square plate pivoting about one corner is another. A grandfather clock’s physical pendulum has a large disk fastened near the bottom of a long rod. In each of these cases, as long as you can figure out the rotational inertia $I$, you can find the angular velocity $\omega$ and the frequency $f$.

### 2.11 Energy in a pendulum

For all oscillating mechanical systems, the total energy is the sum of kinetic and potential energies. In a spring/mass system, the potential energy is that of the spring: $(1/2)kx^2$. With a pendulum, the potential energy is gravitational in origin.

Let us use the bottom of the pendulum’s swing as our reference point for gravitational potential energy: $y = 0$ when $\theta = 0$. Then $PE = mgh$ where $h$ is the height of the bob above our reference. Referring to the diagram, we see that $h = \ell(1 - \cos \theta)$. For a pendulum of total energy $E$, then, we may write

$$E = \frac{1}{2}mv^2 + mg\ell(1 - \cos \theta)$$

**Example.** Suppose we have a simple pendulum with string length $\ell = 1.0$ m, bob mass $0.20$ kg. When the bob is at the bottom of the swing, its velocity is $1.50$ m/s. What is the maximum angle to which the pendulum swings? (That is, what is the amplitude of $\theta$?)

To solve this, we first find the total energy. This will allow us to find the maximum value of $h$, from which we can find the maximum value of $\theta$. The total energy is equal to the kinetic energy at the bottom of the swing:

$$E = \frac{1}{2}mv^2 = \frac{1}{2}(0.20 \text{ kg})(1.50 \text{ m/s})^2 = 0.225 \text{ J}$$

At the maximum value of $\theta$, all energy is potential, so

$$E = mg\ell(1 - \cos \theta)$$

$$1 - \cos \theta = \frac{E}{mg\ell}$$

$$1 - \cos \theta = \frac{0.225 \text{ J}}{(0.20 \text{ kg})(9.8 \text{ m/s}^2)(1.0 \text{ m})} \approx 0.1148$$

$$\cos \theta = 1 - 0.1148 = 0.8852$$

$$\theta = 28^\circ$$

We might note that this angle is quite a bit more than $10^\circ$, so the small-angle approximation does not apply. That means that the restoring force (or torque) is not linearly proportional
to the angle, and this pendulum would not oscillate with purely simple harmonic motion. It would be pretty close, but it’s graph of position vs. time would not be a “good” sine wave. Even so, our calculation is still correct: our energy calculations do not depend on the small-angle approximation.

2.12 Graphing simple harmonic motion

We already know that the general equation for simple harmonic motion is

$$x = A\sin(\omega t)$$

(24)

where $A$ is the amplitude. Further, we know, from a previous argument, that the maximum velocity of the oscillating mass is $A\omega$. We might then expect that the velocity also obeys a “sine wave” shape, with an amplitude of $A\omega$. However, it cannot be the same as the displacement, since the displacement’s maximum ($x = A$) occurs at a time the velocity is at a minimum (zero). So let’s go back to our knowledge from early in first term physics: the velocity is the slope of the position graph. Plotting the sine wave and taking the slope at a few points, we find indeed that we obtain a sinusoidal graph, with maxima occurring where the sine wave is zero. This is a cosine wave, and we illustrate it in the next figure. The velocity may be written

$$v = A\omega \cos (\omega t)$$

(25)

Now, what about the acceleration? We have already shown that the maximum of the acceleration is $A\omega^2$. Again, we expect some kind of sinusoidal shape to the graph. Taking the slope of the velocity graph, we do obtain a sinusoidal shape. However, when the displacement is at $+A$, the acceleration is negative, and when $x = -A$, the acceleration is positive. This makes physical sense: the force on the mass is always opposite the sign of the displacement from equilibrium, since the force is always trying to pull the mass back to equilibrium. The acceleration is in the same direction as the net force, so the acceleration’s sign will always be opposite that of $x$. Thus we find that

$$a = -A\omega^2 \sin (\omega t)$$

(26)

These results are plotted on the next page.
Plots of position, velocity, and acceleration for simple harmonic motion.

Here we use $A = 1.0$ m and $\omega = 1.0$ rad/s.

\[ \text{Displacement, m} \]
\[ \text{time, seconds} \]

\[ \text{Velocity, m/s} \]
\[ \text{time, seconds} \]

\[ \text{Acceleration, m/s}^2 \]
\[ \text{time, seconds} \]
2.13 Why is simple harmonic motion important?

Simple harmonic motion occurs whenever the restoring force is a linear function of displacement: that is, when $F \propto x$. This is true for a spring. We found that it is very close to true for a pendulum if the amplitude is small enough. This turns out to be true for almost any system that has a stable equilibrium point.

We know that the potential energy function for a spring is $(1/2)kx^2$, which we found by taking the area under the curve of the force function $F = kx$ (recall, this is force on the spring, so there is no negative sign.) Another way to describe conditions for SHO is that the potential energy function is quadratic: it depends on $x^2$. For example, we have shown that the potential energy function for a simple pendulum of length $\ell$ is $mg\ell(1 - \cos \theta)$. The cosine function can be described as an infinite series: $\cos \theta = 1 - \theta^2/2! + \theta^4/4! - \ldots$. For small angles, the terms in $\theta^4$ and above are so small that we can ignore them. Plugging $(1 - \theta^2/2)$ into the potential energy function $mg\ell(1 - \cos \theta)$ gives us

$$PE = \frac{1}{2}mg\ell \theta^2$$

This is a quadratic function and looks a lot like $PE = (1/2)kx^2$. So a simple pendulum has the required quadratic PE for small displacements.

For an example less prosaic than a pendulum, let’s look at the covalent bond between hydrogen and chloride in HCl. It can be described by what is called a Lenard-Jones potential. This potential has terms in $1/r^6$ and $1/r^{12}$, where $r$ is the distance between nuclei. It is a strange function indeed. We sketch the function at right. At the equilibrium point $r_o$, the potential energy is a minimum and the force is zero. The force increases for displacements $(r - r_o)$, away from the equilibrium point.

The function itself is not quadratic in $r - r_o$, but something interesting occurs if we consider only small distances from the equilibrium point: the function is approximately proportional to $(r - r_o)^2$. That is, if you “blew up” the graph and looked at the region near the bottom of the potential energy function, it is nearly a parabola: it is almost quadratic. That means that for small oscillations, the system should exhibit simple harmonic motion. Thus, we could use a “masses-on-a-spring” model for the molecule.

Now, we can’t really treat a molecule like this as little hard balls hooked by a spring. Objects that small obey the rules of quantum mechanics, not Newtonian mechanics. (We will get to quantum theory Spring term.) But there are some aspects of the behavior of such molecules that do remind us of simple harmonic motion. The point of this discussion is that any function that has a stable equilibrium point will be quadratic for small enough displacements from equilibrium. Thus, we ought to see something akin to simple harmonic motion showing up in a great many physical systems.