Air Drag Mathematics Reference Guide

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Contents

The purpose of these tutorials is to act as a reference guide for the study of mechanics which involve air drag, which is a critical component of modeling most macroscopic terrestrial motion. These tutorial are particularly suited to the study of sports. This tutorial introduces the following novel ideas:

- The speed of interest for an object is its speed divided by its net terminal speed, even for objects that neither rise nor fall. The relevance of air drag is detemined by "Tovar's Drag Rule" which is introduced.
- Conservation of Quasi-Energy is introduced.

- A new conservation law that applies to 2D motion is introduced.

Tutorial 1

1D Projectile Motion

1.1 Terminal Speed

For one-dimensional projectile motion where the size and speed of the object are such that the quadradic-velocity model dominates,

$$\frac{dv_y}{dt} = a_y = -g - \frac{1}{2m} C_d \rho_{air} A v_y |v_y| \tag{1.1}$$

$$\frac{dy}{dt} = v_y \tag{1.2}$$

where C_D is the drag coefficient, A is the cross-sectional area, and ρ_{air} is the air density.

The terminal speed is reached when the acceleration is zero for a falling object. Setting Eq. (1.1) to zero yields

$$\boxed{v_T = \sqrt{\frac{mg}{\frac{1}{2} C_D \rho_{air} A}}} \tag{1.3}$$

If the mass is constant and the area is increase, then the terminal speed decreases and the effect of air drag increases. This is what happens when a skydiver opens her parachute.

The above traditional terminal speed formula can be recast in terms of the object's density. For a sphere this would become $v_T^2 = (2/3C_D)(\rho_{obj}/\rho_{air})gr$. From this new version of the terminal speed formula it can be seen that if the density is constant, then the larger the object, the larger the terminal speed, and the smaller the effect of air drag. So, a large rock suffers less drag than a small rock even though the larger rock has a larger area.

The larger the object, the SMALLER the effect of air resistance (for a given density).
The larger the object, the LARGER the effect of air resistance (for a given mass).

1.1.1 Tovar's Drag Rule

If the maximum speed of an object is less than 15% of its terminal speed, you can ignore Air Drag.

The percent of energy lost due to drag is approximately $v^2/(2v_T^2)$. If $v/v_T = 0.15$, then only 1.1 % of the energy is lost and the effects of air drag may be neglected.

Object	Mass (g)	C_D	Diameter (mm)	v_T (m/s)	v_T (mph)	$v_{typical}(mph)$	Situation
Bullet	9.72 (150 grn)	.295	7.82	106	237	1984	.30-06 gun
Shot Put	7260 (16 lb.)	.5	125	139	311	32	20 m throw
Baseball	145	0.4	75	36.6	81.8	100	HR Hit
Tennis Ball	57	0.5	65			120	Serve
BasketBall	624	0.3	234				Free Throw
Football		0.06					Long Pass
Golf ball*	46	0.5	42	32			Drive
Hail Stone	.48	0.5	10	14			
Rain Drop	.034	0.5	4	9.0			
Car							Hwy. Drive
Object	Mass (kg)	C_D	Area (m^2)	v_T (m/s)	v_T (mph)		
Person (Flat)	82	0.4					
Person (Pointed)	82	1.0					
Cyclist	68 (150 lb)	0.9					

1.2 1D Projectile Motion - Dynamical Equations

1.2.1 General 1D Motion

The governing differential equations are

$$\frac{dv_y}{dt} = -g\left(1 + \frac{v_y|v_y|}{v_T^2}\right) \tag{1.4}$$

$$\frac{dy}{dt} = v_y \tag{1.5}$$

1.2.2 Upward Motion ($v_{y0} > 0$)

The governing differential equations are

$$\frac{dv_y}{dt} = -g\left(1 + \frac{v_y^2}{v_T^2}\right) \tag{1.6}$$

$$\frac{dy}{dt} = v_y \tag{1.7}$$

and their solutions are

$$\boxed{v_y = v_T \frac{\left(\frac{v_{y0}}{v_T}\right) - tan\left(\frac{gt}{v_T}\right)}{1 + \left(\frac{v_{y0}}{v_T}\right) tan\left(\frac{gt}{v_T}\right)}} \tag{1.8}$$

$$\boxed{y(t) = y_0 + \frac{v_T^2}{g} ln\left[\left(\frac{v_{y0}}{v_T}\right) sin\left(\frac{gt}{v_T}\right) + cos\left(\frac{gt}{v_T}\right)\right]} \tag{1.9}$$

1.2.3 Downward Motion ($v_{y0} < 0$)

The governing differential equations are

$$\frac{dv_y}{dt} = -g\left(1 - \frac{v_y^2}{v_T^2}\right) \tag{1.10}$$

$$\frac{dy}{dt} = v_y \tag{1.11}$$

and their solutions are

$$\boxed{v_y = v_T \frac{\left(\frac{v_{y0}}{v_T}\right) - tanh\left(\frac{gt}{v_T}\right)}{1 - \left(\frac{v_{y0}}{v_T}\right) tanh\left(\frac{gt}{v_T}\right)} = v_T \left[\frac{\left(1 + \frac{v_{y0}}{v_T}\right) - \left(1 - \frac{v_{y0}}{v_T}\right) e^{2\frac{gt}{v_T}}}{\left(1 + \frac{v_{y0}}{v_T}\right) + \left(1 - \frac{v_{y0}}{v_T}\right) e^{2\frac{gt}{v_T}}}\right]} \tag{1.12}$$

$$y(t) = y_0 - \frac{v_T^2}{g} ln \left[cosh\left(\frac{gt}{v_T}\right) - \left(\frac{v_{y0}}{v_T}\right) sinh\left(\frac{gt}{v_T}\right)\right] \tag{1.13}$$

which can also be written

$$\boxed{y(t) = y_0 - \frac{v_T^2}{g} ln \left[\left(1 - \frac{v_{y0}}{v_T}\right) \frac{e^{\frac{gt}{v_T}}}{2} - \left(1 + \frac{v_{y0}}{v_T}\right) \frac{e^{-\frac{gt}{v_T}}}{2}\right]} \tag{1.14}$$

1.2.4 Up and Down Motion

From the equations above, it can be seen that the time it takes for a projectile to reach the top of its flight is

$$\boxed{t_{top} = \frac{v_T}{g} tan^{-1}\left(\frac{v_{y0}}{v_T}\right)} \tag{1.15}$$

then for $0 < t < t_{top}$, the velocity of the projectile is given above. When $t = t_{top}$, the projectile reaches a height of

$$\boxed{y_{top} = y_0 + \frac{v_T^2}{2g} ln(1 + v_{y0}^2/v_T^2)} \tag{1.16}$$

and has zero speed. For $t > t_{top}$, the height of the projectile is

$$\boxed{y(t) = y_{top} - \frac{v_T^2}{g} ln \left[cosh\left(\frac{g(t - t_{top})}{v_T}\right)\right] = y_{top} - \frac{v_T^2}{g} ln \left[\frac{e^{\left(\frac{g(t-t_{top})}{v_T}\right)} - e^{-\left(\frac{g(t-t_{top})}{v_T}\right)}}{2}\right]} \tag{1.17}$$

and the velocity is

$$\boxed{v_y = -v_T tanh\left(\frac{g(t - t_{top})}{v_T}\right) = -v_T \frac{e^{\left(\frac{2g(t-t_{top})}{v_T}\right)} - 1}{e^{\left(\frac{2g(t-t_{top})}{v_T}\right)} + 1}} \tag{1.18}$$

1.3 1D Projectile Motion - Conservation of Quasi-Energy

Energy is not conserved. However, one may obtain Conservation Laws relating the position and velocity in a similar fashion to conservation of energy. The results are the laws for Conservation of "Quasi-Energy."

1.3.1 Upward Motion ($v_{y0} > 0$)

$$\boxed{\underbrace{mgy}_{Q.P.E} + \underbrace{\frac{1}{2} m v_T^2 ln\left[1 + (v_y^2/v_T^2)\right]}_{Q.K.E} = \mathcal{E}_0} \tag{1.19}$$

1.3.2 Downward Motion ($v_{y0} < 0$)

The corresponding quasi-energy expression for downward motion can be obtained by replacing v_T^2 with $-v_T^2$:

$$\boxed{\underbrace{mgy}_{Q.P.E} + \underbrace{\frac{1}{2}mv_T^2 ln\left(\frac{1}{1-(v_y^2/v_T^2)}\right)}_{Q.K.E} = \mathcal{E}_0} \tag{1.20}$$

1.3.3 Up and Down Motion

Here we consider a quasi-energy expression for a projectile that is launched upward, and comes down (though the final position can be different than the initial position). First, the projectile goes up, and reaches a maximum position, so that

$$mgy_{top} = mgy_0 + \frac{1}{2}mv_T^2 ln\left[1 + (v_{y0}^2/v_T^2)\right] \tag{1.21}$$

then the projectile falls some distance. Starting with zero velocity at the top, the downward motion equation becomes

$$mgy + \frac{1}{2}mv_T^2 ln\left(\frac{1}{1-(v_y^2/v_T^2)}\right) = mgy_{top} \tag{1.22}$$

Substituting mgy_{top} from one of these into the other yields

$$\boxed{mgy + \frac{1}{2}mv_T^2 ln\left(\frac{1}{1-(v_y^2/v_T^2)}\right) = mgy_0 + \frac{1}{2}mv_T^2 ln\left[1 + (v_{y0}^2/v_T^2)\right]} \tag{1.23}$$

Note that the quas-kinetic energy going up is different than that going down. It is unusual for kinetic energy to depend on the direction of the projectile.

1.4 Fall Time and Fall Distance

The "Fall Time," also called "characteristic time" is given by

$$\boxed{\tau_T \equiv \frac{v_T}{g}} \tag{1.24}$$

An dropped object that falls 3 characteristic times has reached a speed over 99.5% of v_T. Put another way, an object has (essentially) reached terminal speed after a time $3\frac{v_T}{g}$.

After three terminal times, a dropped object falls a distance of approximately

$$\boxed{d_T = 2.31\frac{v_T^2}{g}} \tag{1.25}$$

Thus, if an object falls $2.31\frac{v_T^2}{g}$, it has (basically) reached terminal speed.

Tutorial 2

2D Motion with Other Forces

2.1 1D Sliding Motion and Similar Problems

While the above equations apply to projectiles, they may be easily altered to apply to **any problem involving only constant forces and drag forces.**

2.1.1 Effects of Friction

Consider the following free body diagram of a sliding (or rolling) object (such as a car) that has frictional forces, drag forces, gravitational forces, and contact forces:

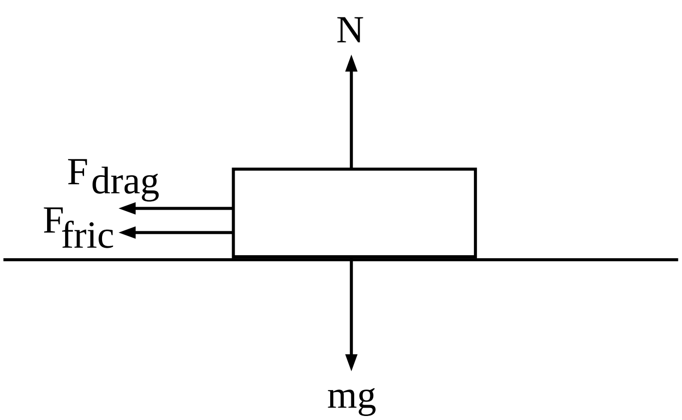


Figure 2.1: A rolling/sliding block experiencing air drag.

The car is moving to the right, and if we define that direction to be positive speed, we must also choose that to be positive acceleration so that Newton's Laws become:

$$N = mg \tag{2.1}$$

$$ma = -F_{fric} - F_{drag} \tag{2.2}$$

These may be rewritten and combined as

$$a = \frac{dv}{dt} = -\mu g - \frac{1}{2m} C_D \rho_{air} A v^2 \tag{2.3}$$

However, if we define

$$\boxed{g' = \mu g} \tag{2.4}$$

$$\boxed{v_T'^2 = \frac{2mg'}{C_D \rho_{air} A}} \tag{2.5}$$

then

$$a = \frac{dv}{dt} = -g' \left(1 + \frac{v^2}{v_T'^2}\right) \tag{2.6}$$

However, this is the same as the "Downward Motion" Eq. (1.10). Thus, we can use all of the Dynamical Equations and the Quasi-Energy Equations for Upward Motion to problems that include buoyancy as long as we replace g and v_T in those equations with g' and v_T' as shown here!

This worked because the frictional force is constant in this problem.

2.1.2 Effects of Buoyancy

Consider the following free body diagram of a projectile that has significant buoyancy forces:

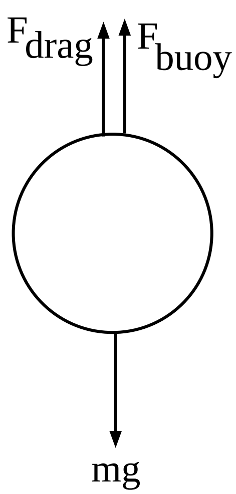


Figure 2.2: An object whose density is near that of air experiences a significant buoyancy force.

For now, suppose the projectile is falling. Newton's Laws reduce to

$$ma = -mg + F_{buoy} + F_{drag} \tag{2.7}$$

This may be rewritten as

$$a = \frac{dv}{dt} = -g + \frac{\rho_{air} V}{m} g + \frac{1}{2m} C_D \rho_{air} A v^2 \tag{2.8}$$

However, if we define

$$\boxed{g' = g - \frac{\rho_{air} V}{m}} \tag{2.9}$$

$$\boxed{v_T'^2 = \frac{2mg'}{C_D \rho_{air} A}} \tag{2.10}$$

then

$$a = \frac{dv}{dt} = -g'\left(1 - \frac{v^2}{v_T'^2}\right) \tag{2.11}$$

However, this is the same as the "Downward Motion" Eq. (1.10). Thus, we can use all of the Dynamical Equations and the Quasi-Energy Equations for Downward Motion to the problem of lateral motion in the presence of friction as long as we replace g and v_T in those equations with g' and v'_T as shown here! For simplicity, we considered Downward Motion. However, the results for **Upward Motion** also apply when the object is going up. In this case, we would use g' and v'_T as in the Downward case. **This worked because the buoyancy force is constant in this problem**. One should notice however that if the object is lighter than air, the g' will be negative.

2.2 2D Quadratic Model

2.2.1 Equations of Motion

Thus, the differential equations of motion are

$$\frac{dv_y}{dt} = -g\left(1 + \frac{v_y}{v_T^2}\sqrt{v_x^2 + v_y^2}\right) \tag{2.12}$$

$$\frac{dv_x}{dt} = -g\left(\frac{v_x}{v_T^2}\sqrt{v_x^2 + v_y^2}\right) \tag{2.13}$$

for projectiles going either up or down.

First we note that these are coupled, nonlinear differential equations. We cannot solve one, and plug the solution into the other, as before. The procedure to solve these time dependent equations is not straightforward. While we can divide these equations by their velocity definitions,

$$\frac{dx}{dt} = v_x \tag{2.14}$$

$$\frac{dy}{dt} = v_y \tag{2.15}$$

in an attempt to obtain a Quasi-Energy expression, the resulting equations do not separate. It is similarly difficult to obtain the corresponding projectile path function.

2.2.2 A Conservation Law

One can not solve the dynamical equations for 2D projectile motion analytically. Similarly, one can not obtain a Quasi-Energy relation. However, one can obtain a relation between speed and angle:

$$\boxed{\frac{v_T^2}{v^2cos^2\theta} + \frac{sin\theta}{cos^2\theta} + ln\left(\frac{1+sin\theta}{cos\theta}\right) = \frac{v_T^2}{v_0^2cos^2\theta_0} + \frac{sin\theta_0}{cos^2\theta_0} + ln\left(\frac{1+sin\theta_0}{cos\theta_0}\right)} \tag{2.16}$$

When the angle θ is zero, the projectile is at the top of it's flight path. From this it follows that the velocity of a projectile at the top of its flight is

$$\frac{v_{x,top}}{v_{x0}} = \frac{1}{\sqrt{1 + \left(\frac{v_{x0}}{v_T}\right)^2\left[\frac{sin\theta_0}{cos^2\theta_0} + ln\left(\frac{1+sin\theta_0}{cos\theta_0}\right)\right]}} \tag{2.17}$$

2.3 EXAMPLES

Example 2.1:

You throw a ball straight up into the air. How long does it take for the ball to return to you? What speed is it going when it returns to you?

Solution 2.1:

It takes a time

$$t_{return} = \left(\frac{v_T}{g}\right)\left[tan^{-1}\left(\frac{v_{y0}}{v_T}\right) + cosh^{-1}\left(\sqrt{1+\frac{v_{y0}}{v_T}}\right)\right] \tag{2.18}$$

to return to you. The projectile has a speed of

$$\boxed{v_y = \frac{v_{y0}}{\sqrt{1+\left(\frac{v_{y0}}{v_T}\right)^2}}} \tag{2.19}$$

♣

Tutorial 3

Appendices

3.1 APPENDIX A: Maple - Solving Transcendental Equations

Many of the above equations are transcendental in a variable of interest, or contain hyperbolic functions. However, solving one equation in one unknown is relatively straightforward in Maple whether hyperbolic functions are included or not. Here is a simple example of how to use the "solve" function in Maple:

$$
\begin{aligned}
&> \quad y := 9;\\
&> \quad ymax := 11;\\
&> \quad x := 15;\\
&> \quad R0 := solve(y = ymax - (4 * ymax/(R \wedge 2)) * (x - R/2) \wedge 2, R);\\
&> \quad R0[2];\\
&> \quad evalf(\%);
\end{aligned}
$$

Maple can do much more. It can plot an implicit equation as a function of an arbitrary variable (Note: If you typed in the above expression into Maple, use the restart command):

$$
\begin{aligned}
&> \quad restart;\\
&> \quad f := solve(v - ln(1 + v) = x - ln(1 + x), v);\\
&> \quad plot(f, x = 0..2);
\end{aligned}
$$

In this particular case there is more than one solution, but Maple finds only the wrong one. It finds $v = x$. To get Maple to find the other solution, change one of the 1's to 1.00000001 like so:

$$> \quad f := solve(v - ln(1 + v) = x - ln(1.0000000001 + x), v);$$

This will not change the result in any practical sense, but the $v = x$ solution is no longer valid, so Maple must try something else.

3.2 APPENDIX B: Drag Graphs for Lookup

Below are some graphs one can use to interpolate the desired information. However, one needs to keep in mind these formulas. The formula for the terminal speed is

$$v_T = \sqrt{\frac{mg}{\frac{1}{2}C_D \rho_{air} A}} \tag{3.1}$$

Some No Drag Formulas:

$$R = \frac{v_0^2}{2g} sin^2(\theta) \tag{3.2}$$

$$y_{max} = \frac{v_0^2}{2g} sin^2(\theta) \tag{3.3}$$

$$T_{flight} = \frac{2v_0}{g} sin(\theta) \tag{3.4}$$

Interpolation may be used in the following graphs to physics problems that include air drag:

1. Optimum Angle ($0 < v0/vT < 1$)
2. Optimum Angle ($v0/vT > 1$)
3. Range - Surface-to-Surface ($0 < v0/vT < 1$)
4. Range - Surface-to-Surface ($v0/vT > 1$)
5. Max. Height Achieved ($0 < v0/vT < 1$)
6. Max. Height Achieved ($v0/vT > 1$)
7. Time of Flight - Surface-to-Surface ($0 < v0/vT < 1$)
8. Time of Flight - Surface-to-Surface ($v0/vT > 1$)

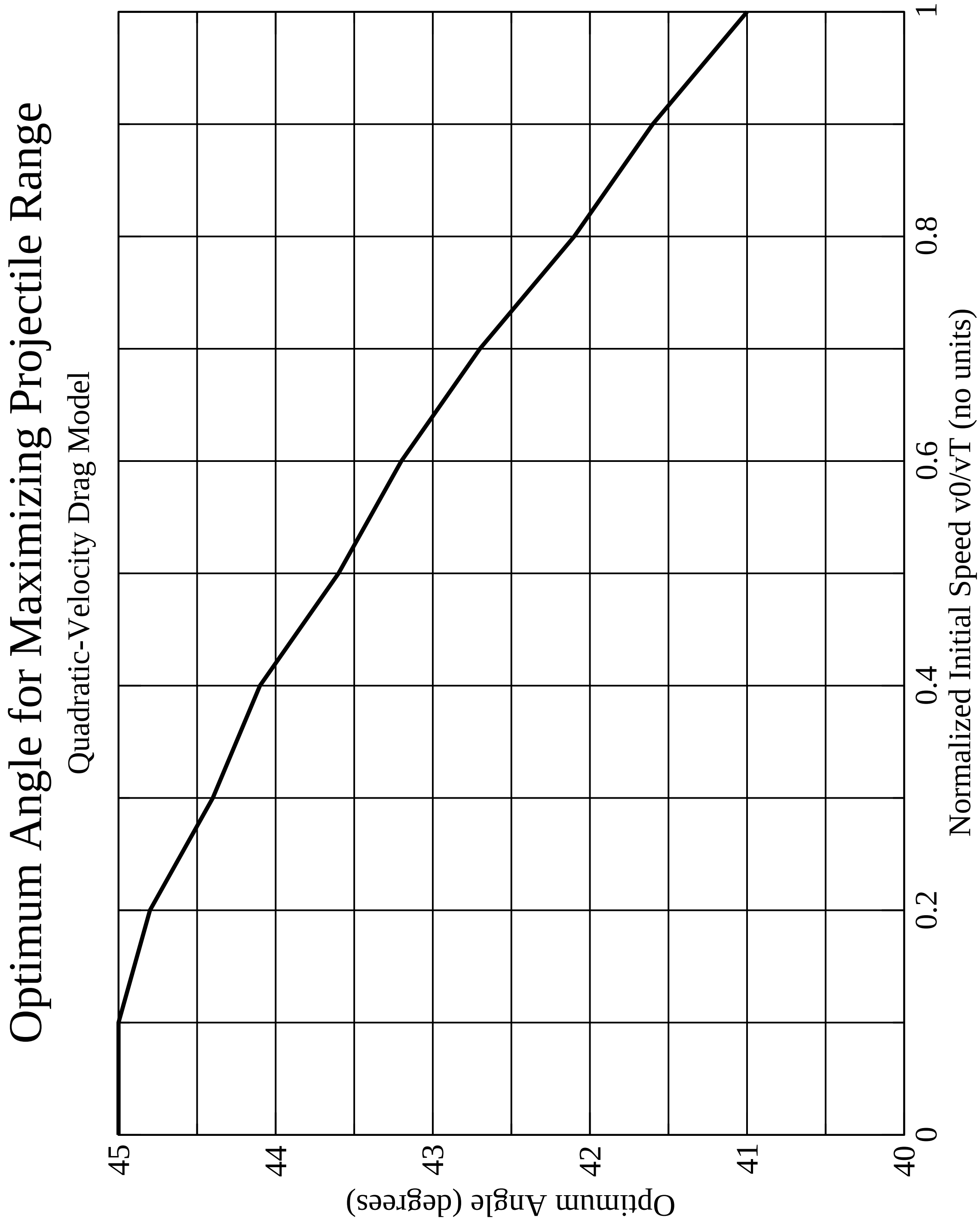


Figure 3.1: Optimum Surface-to-Surface Launch Angle including Drag

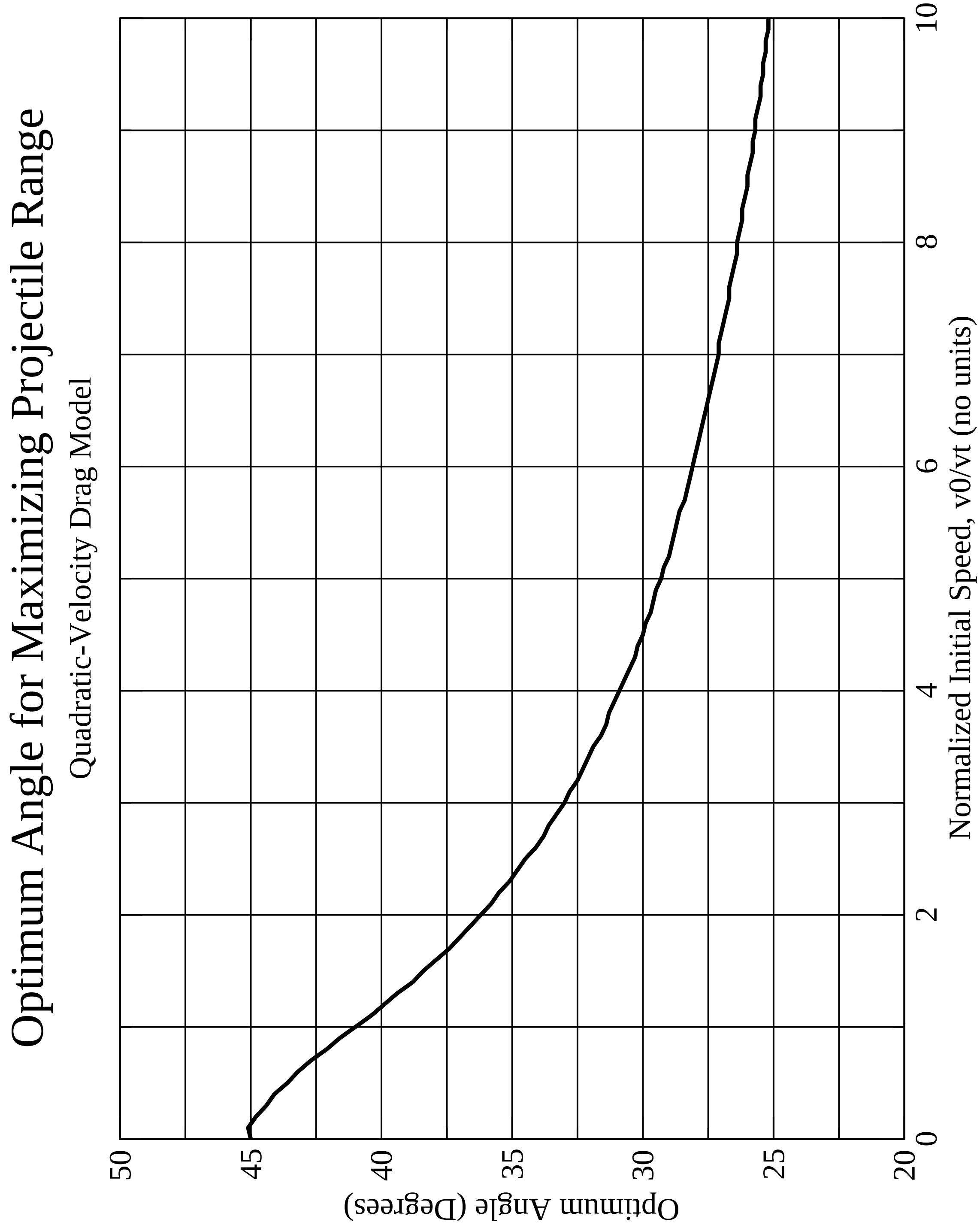


Figure 3.2: Optimum Surface-to-Surface Launch Angle including Drag

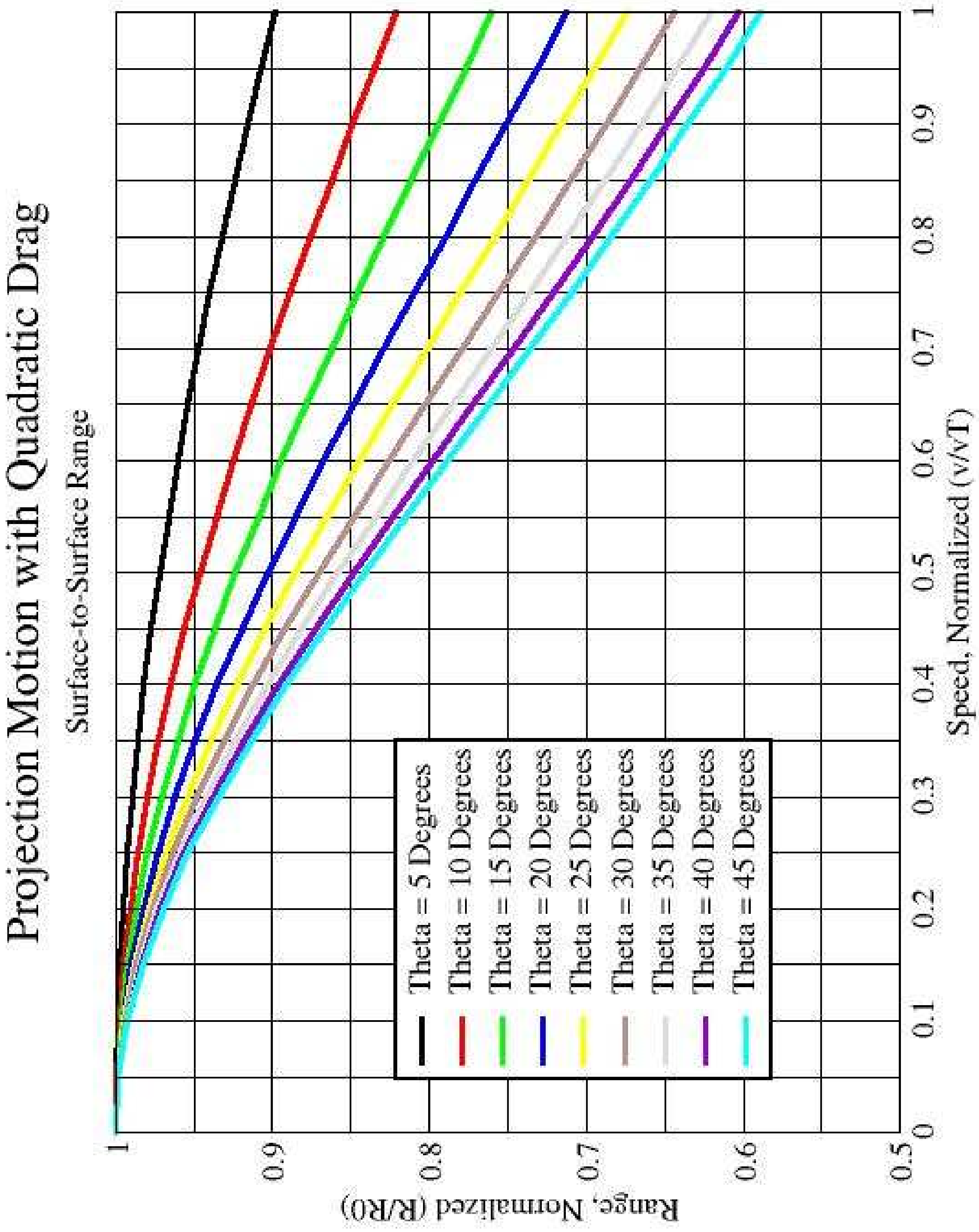


Figure 3.3: Surface-to-Surface Range including Drag

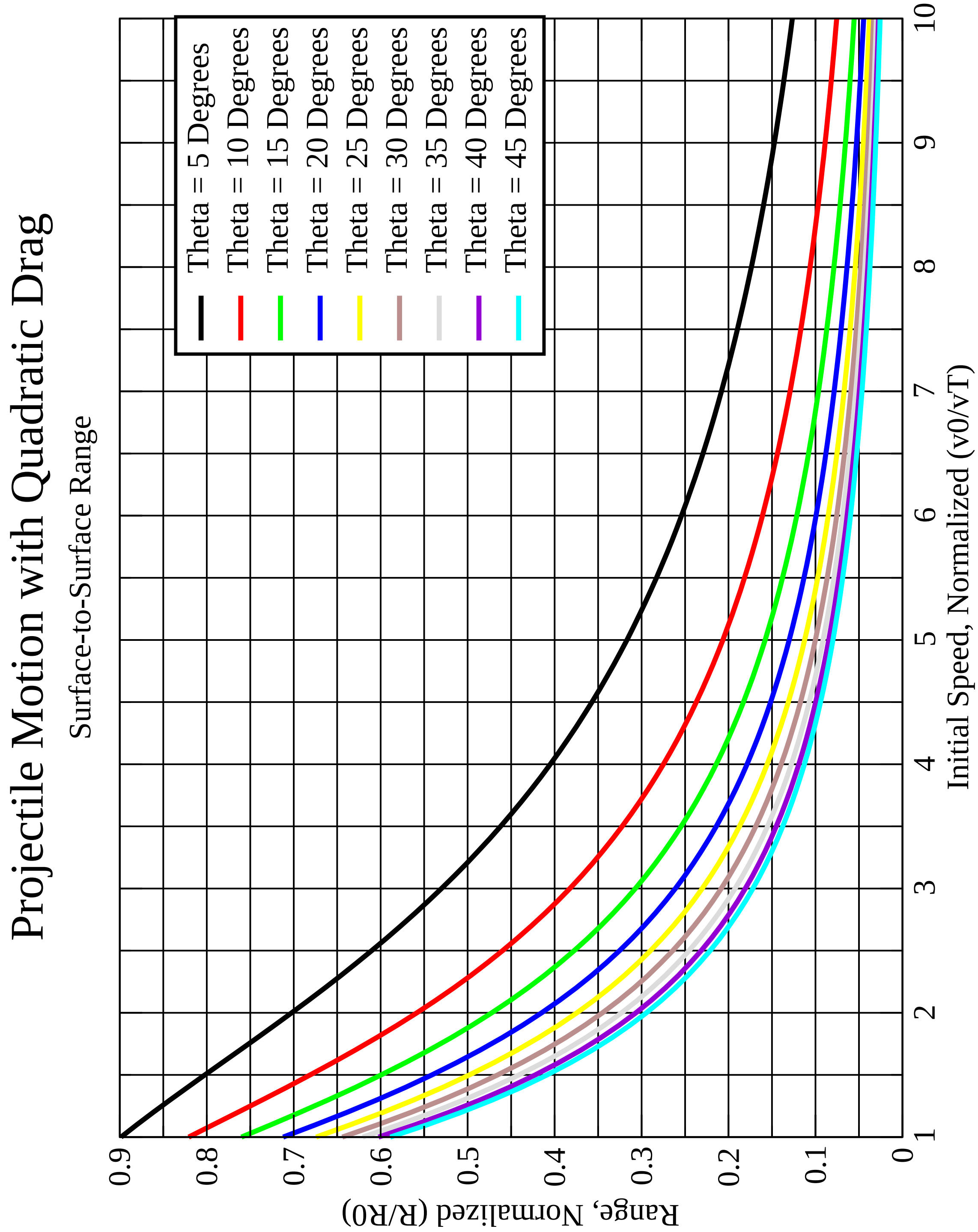


Figure 3.4: Surface-to-Surface Range including Drag

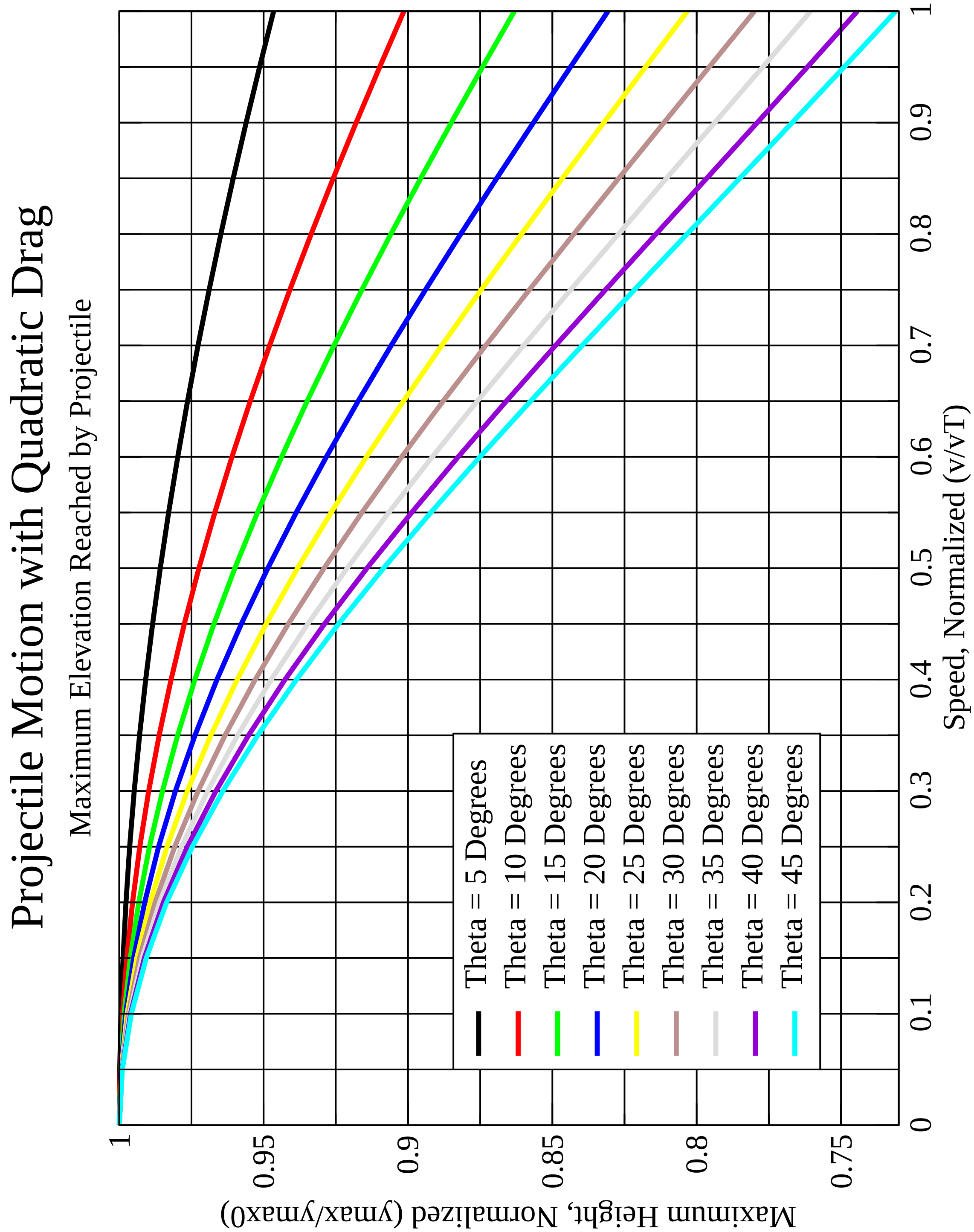


Figure 3.5: Maximum Height of a Projectile including Drag

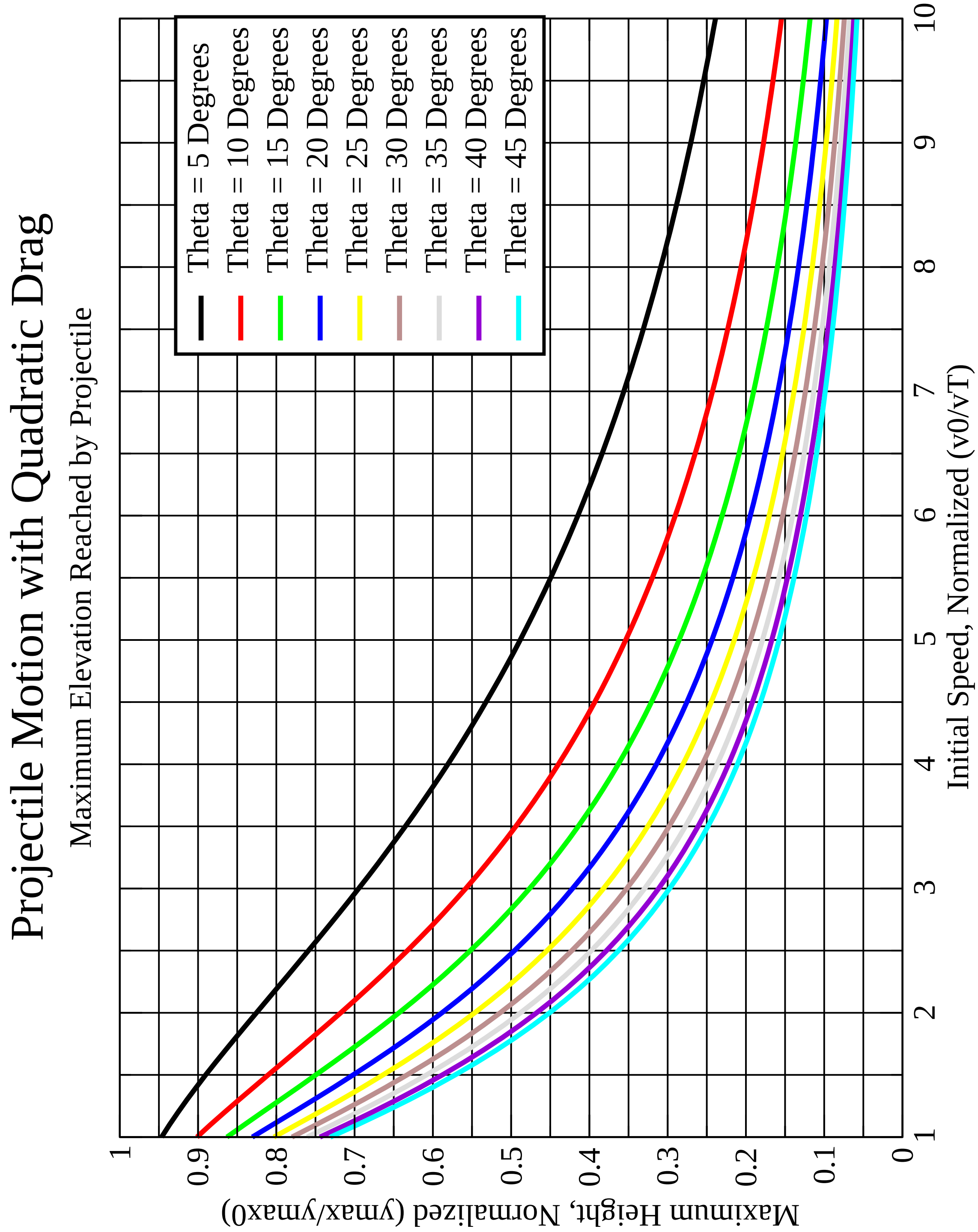


Figure 3.6: Maximum Height of a Projectile including Drag

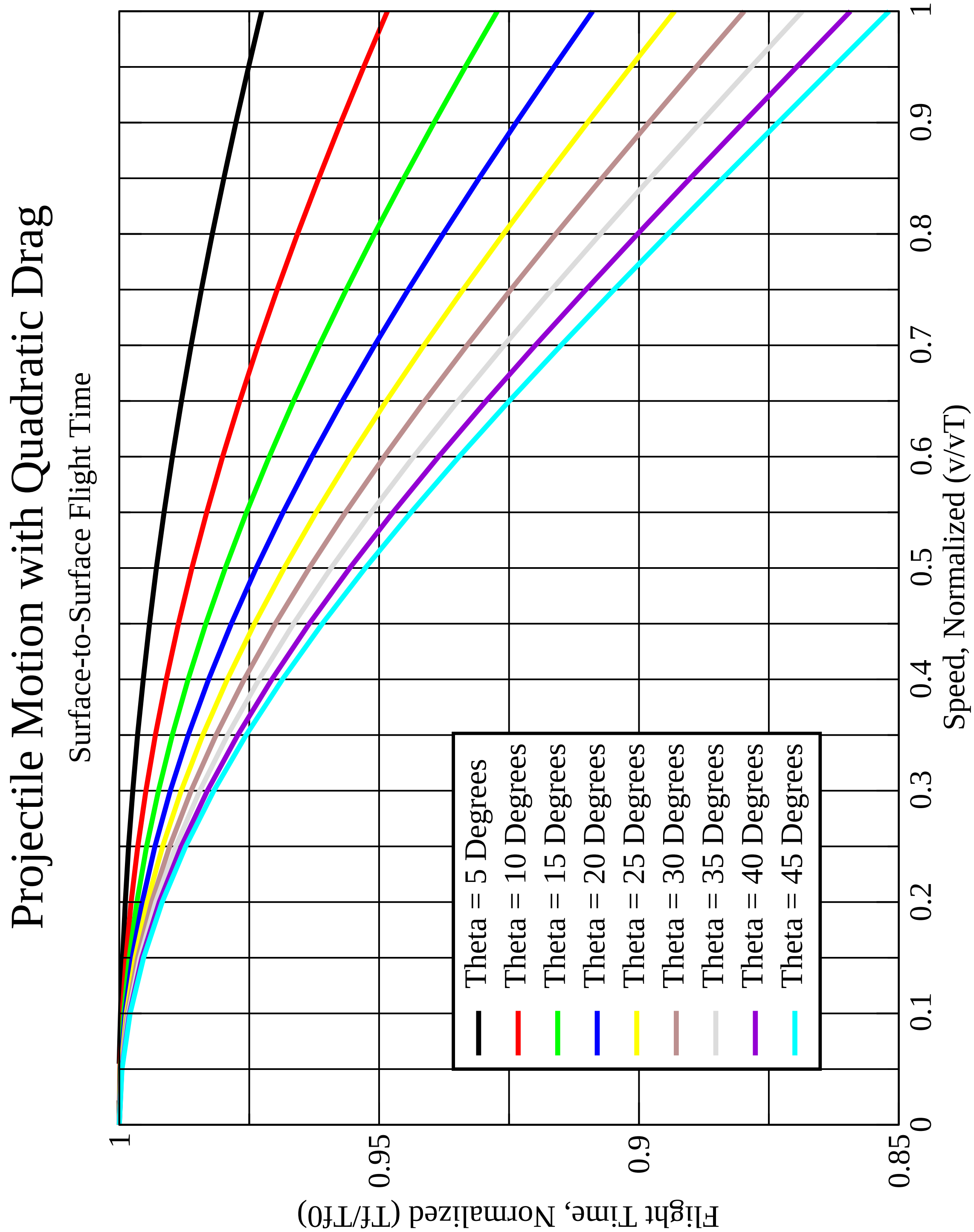


Figure 3.7: Surface-to-Surface Flight Time including Drag

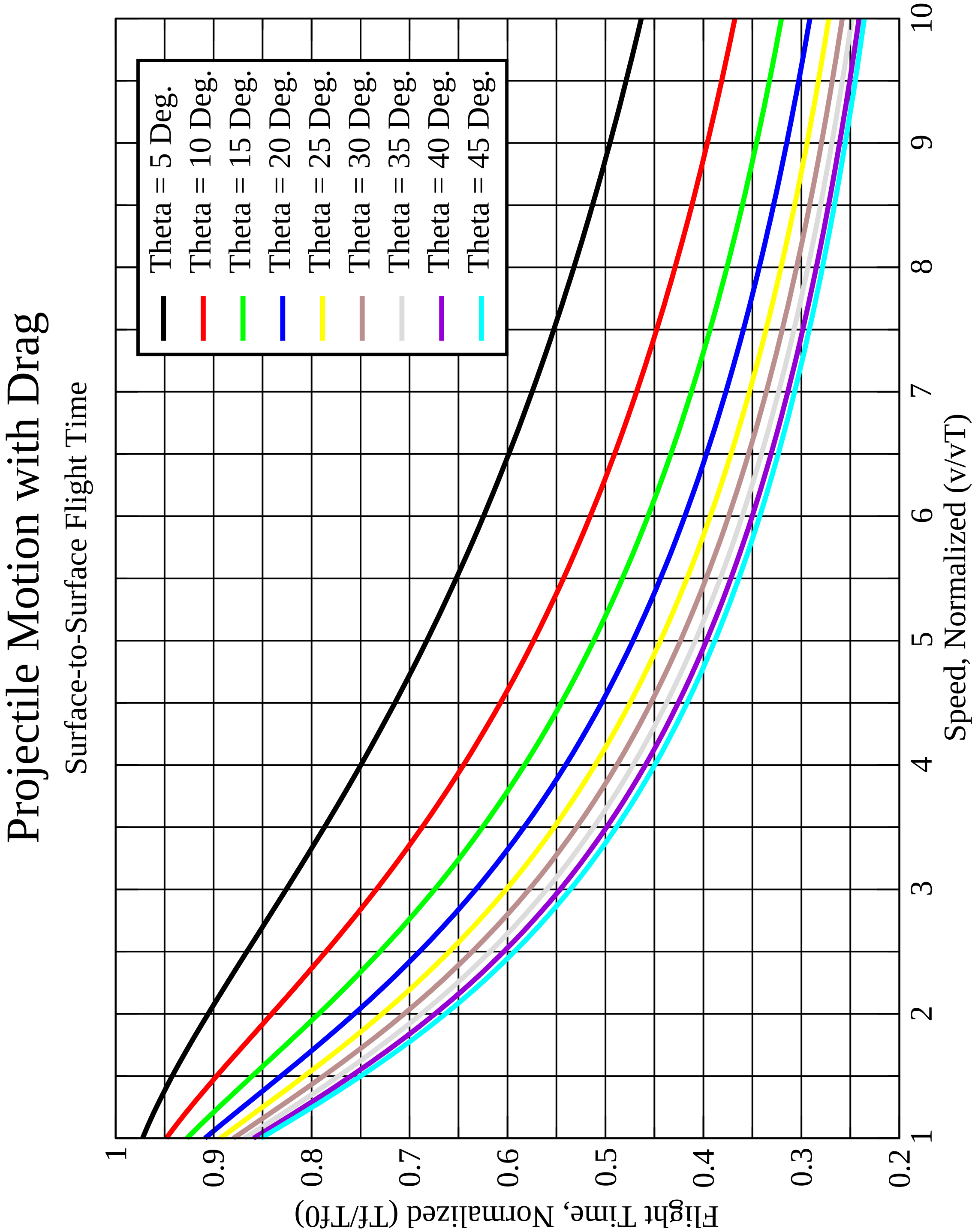


Figure 3.8: Surface-to-Surface Flight Time including Drag

3.3 APPENDIX C: Maple - 2D Projectile Motion

Maple can solve the differential equations governing the 2D motion of a projectile. Here is the first part of a sample worksheet:

Solution to ODEs

Projectile Motion with Drag

```
>
>
>
> restart:
> with(plots):
Warning, the name changecoords has been redefined
```

```
> yacc:=diff(vy(t),t)=-(1+vy(t)*sqrt(vx(t)^2+vy(t)^2));
```

$$yacc := \frac{d}{dt}\,\mathrm{vy}(t) = -1 - \mathrm{vy}(t)\sqrt{\mathrm{vx}(t)^2 + \mathrm{vy}(t)^2}$$

```
>
> xacc:=diff(vx(t),t)=-vx(t)*sqrt(vx(t)^2+vy(t)^2);
```

$$xacc := \frac{d}{dt}\,\mathrm{vx}(t) = -\mathrm{vx}(t)\sqrt{\mathrm{vx}(t)^2 + \mathrm{vy}(t)^2}$$

```
> yvel:=diff(y(t),t)=vy(t);
```

$$yvel := \frac{d}{dt}\,\mathrm{y}(t) = \mathrm{vy}(t)$$

```
> xvel:=diff(x(t),t)=vx(t);
```

$$xvel := \frac{d}{dt}\,\mathrm{x}(t) = \mathrm{vx}(t)$$

```
> init_con:=vy(0)=.80*sin(30*Pi/180),
vx(0)=.80*evalf(cos(30*Pi/180)), y(0)=0, x(0)=0;
```

$$init\_con := \mathrm{vy}(0) = 0.4000000000,\ \mathrm{vx}(0) = 0.6928203232,\ \mathrm{y}(0) = 0,\ \mathrm{x}(0) = 0$$

```
> odesys:=[yacc, xacc, xvel, yvel, init_con];
```

$$odesys := \left[\frac{d}{dt}\,\mathrm{vy}(t) = -1 - \mathrm{vy}(t)\sqrt{\mathrm{vx}(t)^2 + \mathrm{vy}(t)^2},\ \frac{d}{dt}\,\mathrm{vx}(t) = -\mathrm{vx}(t)\sqrt{\mathrm{vx}(t)^2 + \mathrm{vy}(t)^2},\right.$$

$$\frac{d}{dt}\,\mathrm{x}(t) = \mathrm{vx}(t),\ \frac{d}{dt}\,\mathrm{y}(t) = \mathrm{vy}(t),\ \mathrm{vy}(0) = 0.4000000000,\ \mathrm{vx}(0) = 0.6928203232,$$

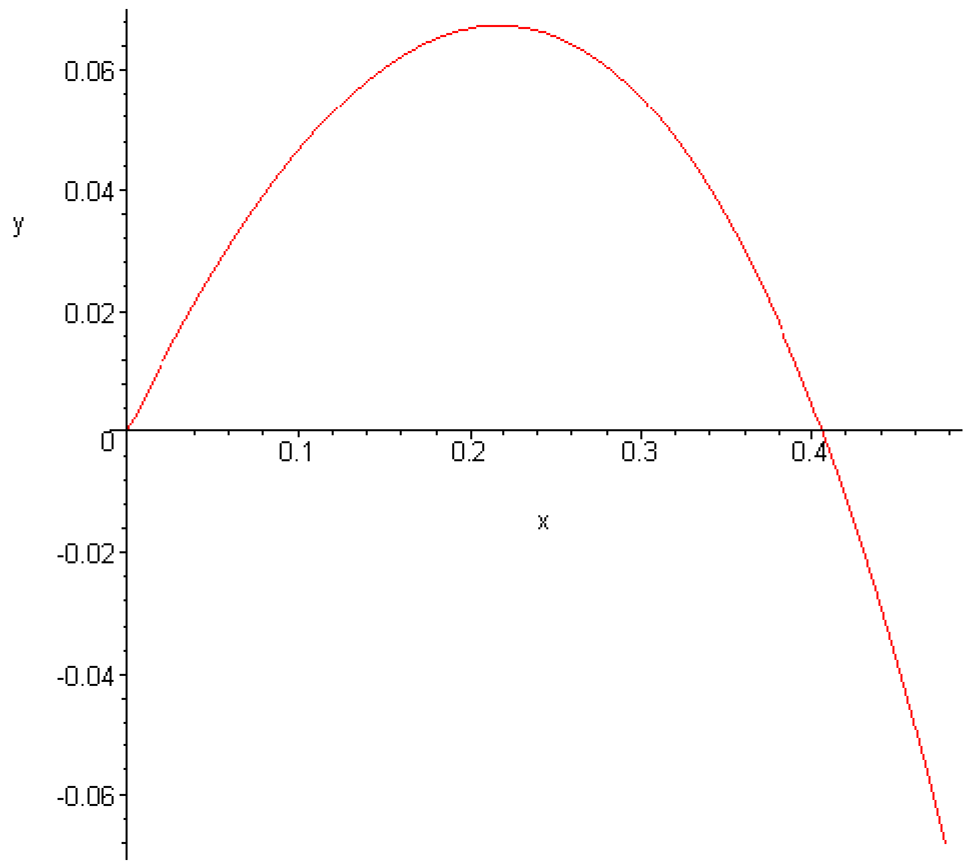
$$\left.\mathrm{y}(0) = 0,\ \mathrm{x}(0) = 0\right]$$

```
> dsol:=dsolve(odesys, numeric);
```

$$dsol := \mathbf{proc}(x\_rkf45)\ \ldots\ \mathbf{end\ proc}$$

```
> odeplot(dsol,[x(t),y(t)],0..0.9);
```

Here is the graph produced and the second part of the worksheet:




```
> dsol(0.3);
    [t =۰.۳, vx(t) =۰.۵۶۷۱۷۸۵۴۲۰۸۷۳۳۳۶۳۶, vy(t) =۰.۰۵۴۰۰۹۶۳۳۹۷۰۸۹۳۷۶۹۰,
        x(t) =۰.۱۸۷۴۹۱۷۵۷۸۵۰۶۶۱۳۹۴, y(t) =-۰.۰۶۵۹۷۵۴۴۳۸۳۵۴۸۲۰۱۹۸ ]

>
```

3.4 APPENDIX D: Spreadsheets - Solving ODEs using RK2

Shown is a spreadsheet that solves the following differential equation using Second Order Runge-Kutta:

$$\frac{dy}{dt} = -10y \tag{3.5}$$

			k1	Euler				RK2	
t	y	f(yi)	dt*f(yi)	yi+ dt*f(yi)	(k1)/2	yi+(k1)/2	f(yi+(k1)/2)	yi+dt*f(yi+(k1)/2)	Actual
0	100			100				100	100
0.01	90.50	-1000.00	-10.00	90.00	-5.00	95.00	-950.00	90.50	90.48
0.02	81.90	-905.00	-9.05	81.45	-4.53	85.98	-859.75	81.90	81.87
0.03	74.12	-819.03	-8.19	73.71	-4.10	77.81	-778.07	74.12	74.08
0.04	67.08	-741.22	-7.41	66.71	-3.71	70.42	-704.16	67.08	67.03
0.05	60.71	-670.80	-6.71	60.37	-3.35	63.73	-637.26	60.71	60.65
0.06	54.94	-607.08	-6.07	54.64	-3.04	57.67	-576.72	54.94	54.88
0.07	49.72	-549.40	-5.49	49.45	-2.75	52.19	-521.93	49.72	49.66
0.08	45.00	-497.21	-4.97	44.75	-2.49	47.23	-472.35	45.00	44.93
0.09	40.72	-449.98	-4.50	40.50	-2.25	42.75	-427.48	40.72	40.66
0.10	36.85	-407.23	-4.07	36.65	-2.04	38.69	-386.87	36.85	36.79
0.11	33.35	-368.54	-3.69	33.17	-1.84	35.01	-350.11	33.35	33.29
0.12	30.18	-333.53	-3.34	30.02	-1.67	31.69	-316.85	30.18	30.12
0.13	27.32	-301.84	-3.02	27.17	-1.51	28.68	-286.75	27.32	27.25
0.14	24.72	-273.17	-2.73	24.59	-1.37	25.95	-259.51	24.72	24.66
0.15	22.37	-247.22	-2.47	22.25	-1.24	23.49	-234.86	22.37	22.31
0.16	20.25	-223.73	-2.24	20.14	-1.12	21.25	-212.55	20.25	20.19
0.17	18.32	-202.48	-2.02	18.22	-1.01	19.24	-192.35	18.32	18.27
0.18	16.58	-183.24	-1.83	16.49	-0.92	17.41	-174.08	16.58	16.53
0.19	15.01	-165.83	-1.66	14.93	-0.83	15.75	-157.54	15.01	14.96
0.20	13.58	-150.08	-1.50	13.51	-0.75	14.26	-142.58	13.58	13.53
0.21	12.29	-135.82	-1.36	12.22	-0.68	12.90	-129.03	12.29	12.25
0.22	11.12	-122.92	-1.23	11.06	-0.61	11.68	-116.77	11.12	11.08
0.23	10.07	-111.24	-1.11	10.01	-0.56	10.57	-105.68	10.07	10.03
0.24	9.11	-100.67	-1.01	9.06	-0.50	9.56	-95.64	9.11	9.07
0.25	8.25	-91.11	-0.91	8.20	-0.46	8.66	-86.55	8.25	8.21
0.26	7.46	-82.45	-0.82	7.42	-0.41	7.83	-78.33	7.46	7.43
0.27	6.75	-74.62	-0.75	6.72	-0.37	7.09	-70.89	6.75	6.72
0.28	6.11	-67.53	-0.68	6.08	-0.34	6.42	-64.16	6.11	6.08
0.29	5.53	-61.12	-0.61	5.50	-0.31	5.81	-58.06	5.53	5.50
0.30	5.01	-55.31	-0.55	4.98	-0.28	5.25	-52.55	5.01	4.98

		k1		Euler				RK2
t	y	f(yi)	dt*f(yi)	yi+ dt*f(yi)	(k1)/2	yi+(k1)/2	f(yi+(k1)/2)	yi+dt*f(yi+(k1)/2)
0	100			100				100
0.01	90.50	-1000.00	-10.00	90.00	-5.00	95.00	-950.00	90.50
0.02	81.90	-905.00	-9.05	81.45	-4.53	85.98	-859.75	81.90
0.03	74.12	-819.03	-8.19	73.71	-4.10	77.81	-778.07	74.12
0.04	67.08	-741.22	-7.41	66.71	-3.71	70.42	-704.16	67.08
0.05	60.71	-670.80	-6.71	60.37	-3.35	63.73	-637.26	60.71
0.06	54.94	-607.08	-6.07	54.64	-3.04	57.67	-576.72	54.94
0.07	49.72	-549.40	-5.49	49.45	-2.75	52.19	-521.93	49.72
0.08	45.00	-497.21	-4.97	44.75	-2.49	47.23	-472.35	45.00
0.09	40.72	-449.98	-4.50	40.50	-2.25	42.75	-427.48	40.72
0.10	36.85	-407.23	-4.07	36.65	-2.04	38.69	-386.87	36.85
0.11	33.35	-368.54	-3.69	33.17	-1.84	35.01	-350.11	33.35
0.12	30.18	-333.53	-3.34	30.02	-1.67	31.69	-316.85	30.18
0.13	27.32	-301.84	-3.02	27.17	-1.51	28.68	-286.75	27.32
0.14	24.72	-273.17	-2.73	24.59	-1.37	25.95	-259.51	24.72
0.15	22.37	-247.22	-2.47	22.25	-1.24	23.49	-234.86	22.37
0.16	20.25	-223.73	-2.24	20.14	-1.12	21.25	-212.55	20.25
0.17	18.32	-202.48	-2.02	18.22	-1.01	19.24	-192.35	18.32
0.18	16.58	-183.24	-1.83	16.49	-0.92	17.41	-174.08	16.58
0.19	15.01	-165.83	-1.66	14.93	-0.83	15.75	-157.54	15.01
0.20	13.58	-150.08	-1.50	13.51	-0.75	14.26	-142.58	13.58
0.21	12.29	-135.82	-1.36	12.22	-0.68	12.90	-129.03	12.29
0.22	11.12	-122.92	-1.23	11.06	-0.61	11.68	-116.77	11.12
0.23	10.07	-111.24	-1.11	10.01	-0.56	10.57	-105.68	10.07
0.24	9.11	-100.67	-1.01	9.06	-0.50	9.56	-95.64	9.11
0.25	8.25	-91.11	-0.91	8.20	-0.46	8.66	-86.55	8.25
0.26	7.46	-82.45	-0.82	7.42	-0.41	7.83	-78.33	7.46
0.27	6.75	-74.62	-0.75	6.72	-0.37	7.09	-70.89	6.75
0.28	6.11	-67.53	-0.68	6.08	-0.34	6.42	-64.16	6.11
0.29	5.53	-61.12	-0.61	5.50	-0.31	5.81	-58.06	5.53
0.30	5.01	-55.31	-0.55	4.98	-0.28	5.25	-52.55	5.01

3.5 APPENDIX E: Hyperbolic Sinusoidal Functions

Maple and Spreadsheet programs recognize hyperbolic sinusoidal functions. Though their definitions are sometime couched in terms of complex functions, they are real (if real numbers are put into them).

$$cosh(x) = \frac{e^x + e^{-x}}{2} = cos(ix) = cosh(-x) \tag{3.6}$$

$$sinh(x) = \frac{e^x - e^{-x}}{2} = -isin(ix) = -sinh(-x) \tag{3.7}$$

$$tanh(x) = \frac{sinh(x)}{cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = -itan(ix) = -tanh(-x) \tag{3.8}$$

The hyperbolic sinusoidal functions are similar to their sinusoidal counterparts in the sense that there exist parallel formulas such as addition formulas and Pythagorean Formula:

$$cosh^2(x) - sinh^2(x) = 1 \tag{3.9}$$

Below are the graphs for the hyperbolic trigonometric functions.

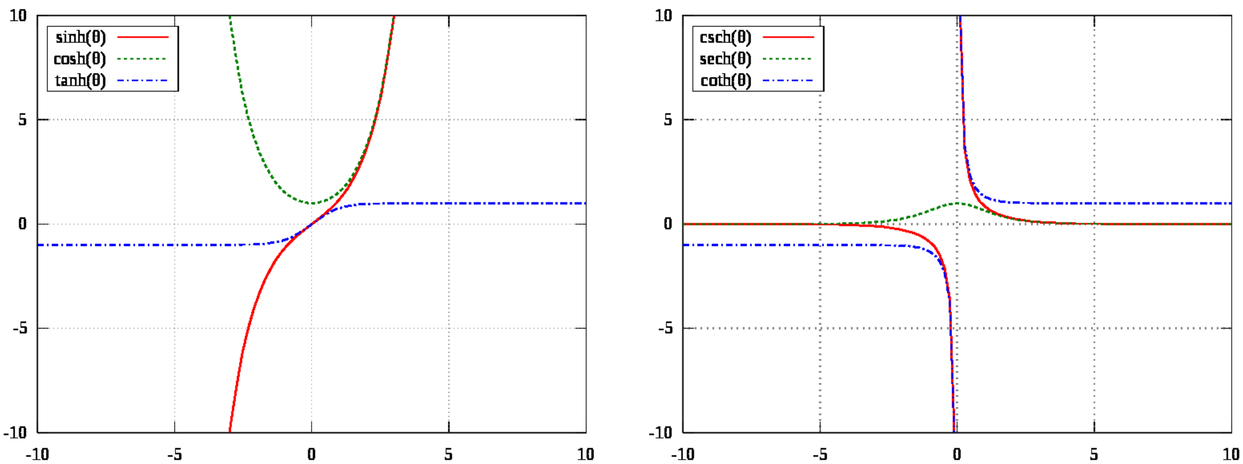


Figure 3.9: cosh=parabolic-looking-thing, sinh=x^3-looking-thing