Trigonometry Explained

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Tutorial 1

Trigonometry

1.1 Introduction

The goal of this tutorial is to explain how trigonometry could be developed by simply drawing some triangles. The reader is lead into "inventing" trigonometry - not just the HOWs, but the WHYs as well. During course work, it is easy to "miss the forest from the trees." Here we look at the forest, and maybe a tree or two. This tutorial is NOT meant to substitute for course work because learning mathematics is like riding a bicycle - you learn it by doing it (as opposed to just reading about it). This tutorial is also very different than many reviews that merely state results. The emphasis here is on the mental process of developing the mathematics. It is hoped that the reader will gain a sense of discovery.

This tutorial is intended for anyone who knows a little algebra, and wants to learn what trigonometry is about. This may include students who just finished a course in algebra, students who recently completed a course in trigonometry but didn't quite get the "big picture," students who had trigonometry years ago and want a review, and educators (including those who homeschool). This tutorial may also be useful to students in the middle of a trigonometry course because the best way to learn mathematics (and science, in general) is to examine the trees for a little while, then the forest, then the trees again, then the forest again... This zooming in and zooming out helps retention immeasurably.

While going through this tutorial, the reader should have a pencil and a calculator handy - learning is not a spectator sport!

1.2 Why Trigonometry?

As can be seen from the first three figures...

- irregular polygons can be made up of many triangles.
- regular polygons can be made up of many triangles.
- Irregular triangles can be made up of right triangles.

A right triangle is one in which two of the sides are at 90° (or perpendicular) to each other, as shown in Fig. (1.4).

Basically, we cannot reduce right triangles into simpler figures. Therefore, if we want to learn about the geometry of figures, we need to study right triangles *first*. Once we have learned about the relationships between the sides and the angles of right triangles, we can apply the results to more complicated (including multidimensional) figures.[1]

[1]What about smooth figures, such as circles? It ends up that we can break those into a large number (infinite, actually) of right triangles and rectangles. Indeed, Archimedes did this problem at about 260 B.C. to help him determine π. It may or may

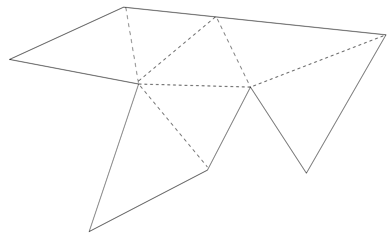


Figure 1.1: An irregularly shaped object with many sides can be divided into triangles.

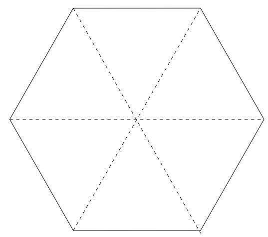


Figure 1.2: A regularly shaped object with many sides can be divided into triangles.

This is a common technique in math and science. To understand complicated ideas and structures, we start by investigating fundamental "building blocks." We can then apply knowledge gained to help us understand the more complicated goings-on in the natural world.

1.3 The Pythagorean Theorem

The first thing we need to known about right triangles is that the sides of a right triangle (a, b, and c) are not independent. Given any two, one can calculate the third. This should not be a big surprise. Draw two sides of a right triangle, and you have no choice as to the length of the third line you draw. This basic result is known as the **Pythagorean Theorem**:

$$c^2 = a^2 + b^2 \tag{1.1}$$

where a, b, and c are as labeled on Fig. 1.4. We could determine this empirically by merely drawing lots of right triangles and measuring their sides.[2] There are many geometrical proofs of the theorem[3], but they are

not sound too bad, but try doing it using archaic notation such as Roman numerals!

[2]This would make a good laboratory exercise.

[3]For example, do a World Wide Web search for "proof Pythagorean theorem."

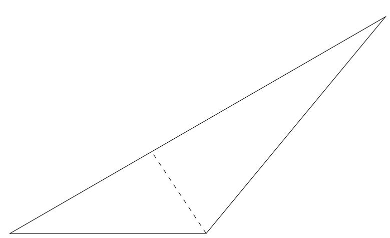


Figure 1.3: An arbitrary triangle can be divided into two right triangles.

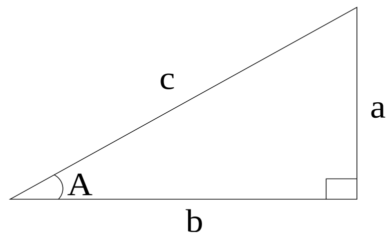


Figure 1.4: A right triangle has a 90° angle. The longest side, c, is called the hypotenuse.

not of interest here.

We may note that if the sides of a right triangle are all whole numbers, then the lengths of these sides are called **Pythagorean triplets**:
$3^2 + 4^2 = 5^2$
$5^2 + 12^2 = 13^2$
$6^2 + 8^2 = 10^2$
$7^2 + 24^2 = 25^2$
$8^2 + 15^2 = 17^2$
$9^2 + 12^2 = 15^2$
$9^2 + 40^2 = 41^2$
$10^2 + 24^2 = 26^2$
$11^2 + 60^2 = 61^2$
$12^2 + 16^2 = 20^2$
$12^2 + 35^2 = 37^2$
$13^2 + 84^2 = 85^2$
A computer program was written to generate these triplets.[4]

Dividing both sides of the Pythagorean Theorem by c yields

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1 \tag{1.2}$$

This equation reemphasizes that there are only two independent factors of a right triangle. It also suggests that we look at the ratios of the sides, which we'll do in the next section.

Application 1.1:

Construction

Before laying a concrete foundation, construction workers make string lines to determine where their wooden "forms" should go. To insure that their string lines (and hence their forms) are at exactly 90°, they measure 3 feet from a corner and make a mark on the string. On the crossing string they make a mark 4 feet from the corner. If the distance between the two marks is exactly 5 feet, then the corner is "square."

They are really using the 3-4-5 Pythagorean triplet. They also make extensive use of the 6-8-10 triplet. They tend to not use the 5-12-13 triplet (nor do they tend to know about that one) because you want a triangle that has angles closer to 45°. ♣

1.4 Trigonometry: the Main Idea

Now, instead of looking at arbitrary right triangles, let's examine right triangles that share a common angle. These right triangles are called **similar triangles**:

[4] Can you figure out what the next triplet is? Answer at end of tutorial. Hint: it starts with 14^2+.

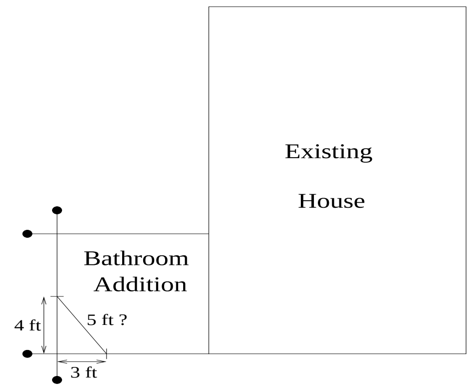


Figure 1.5: Pythagorean Triplets are used to determine if two strings are at right angles to each other.

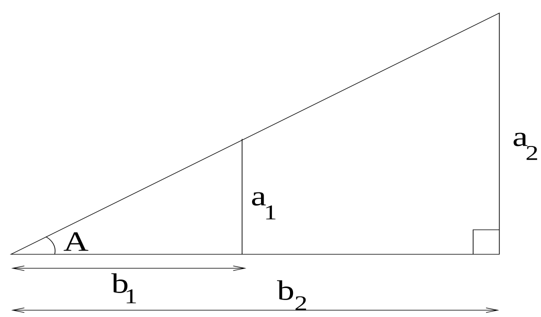


Figure 1.6: Similar triangles.

Again, we can draw pairs of similar triangles, and start measuring their sides. If we do this we would immediately see an obvious pattern.[5] This pattern is the **law of similar triangles**:

> **For a right triangle with a given angle, A, the ratio of any two sides is constant independent of the size of the triangle.**

For example, in Fig. 1.6

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} \tag{1.3}$$

This ratio is the same for all right triangles which have the same angle, A, associated with them. Of course this ratio is different for different values of A. Thus, there is a function that specifies this ratio for different angles - it is called the tangent function, or $tan(A)$.

We can similarly define the sine and cosine functions:

$$sin(A) = \frac{a}{c} = \frac{opposite}{hypotenuse} \tag{1.4}$$

$$cos(A) = \frac{b}{c} = \frac{adjacent}{hypotenuse} \tag{1.5}$$

$$tan(A) = \frac{a}{b} = \frac{opposite}{adjacent} = \frac{sin(A)}{cos(A)} \tag{1.6}$$

[5]This too would make a good laboratory exercise.

where, as before, a, b, and c are defined as in Fig. (1.4). A memory device sometimes used to for these relationships involves the (fictional) American Indian chief **SOH CAH TOA**:

Sine is **O**pposite over **H**ypotenuse.
Cosine is **A**djacent over **H**ypotenuse.
Tangent is **O**pposite over **A**djacent.

With three sides, there are six possible ratios (OH, OA, AH, AO, HA, HO) of the sides of a right triangle. So, we further define the cosecant, secant, and cotangent functions as

$$csc(A) = \frac{1}{sin(A)} \tag{1.7}$$

$$sec(A) = \frac{1}{cos(A)} \tag{1.8}$$

$$cot(A) = \frac{1}{tan(A)} \tag{1.9}$$

Now, the ratio of any two sides has a definition.

Inclines are often encountered in everyday life. The slope of an incline is, of course, rise/run. However, this is precisely how the tangent function is defined. The percent grade of an incline is just the slope measured in percent, so that

$$slope = percent\ grade = tan(\theta) \tag{1.10}$$

For example, a 20% grade has a slope of 0.2 and an angle of elevation of $tan^{-1}(0.2) = 11.3°$.

Application 1.2:

The Height of a Pyramid

Thales of Miletus (624-547 B.C.), often regarded as the first physicist, was asked to measure the height of a pyramid. Since there is no vertical surface on a pyramid, he couldn't simply measure it. Furthermore, trigonometry hadn't been invented yet. However, he did have a notion of similar triangles, which is all he needed. He basically stuck a stick in the ground and measured its height

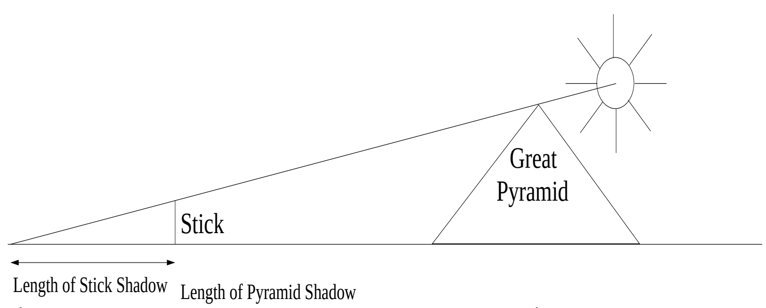


Figure 1.7: Thales measured the height of a pyramid using similar triangles.

and the length of the shadow it gave off. He then measured the length of the shadow of the pyramid. Using the law of similar triangles,

$$\frac{Height\ of\ Stick}{Length\ of\ Stick\ Shadow} = \frac{Height\ of\ Pyramid}{Length\ of\ Pyramid\ Shadow} \tag{1.11}$$

From this expression it is easy to determine the height of the pyramid. ♣

1.5 Degrees and Radians

One can measure angles in degrees or radians. Degrees exist in everyday usage, but not all are familiar with radians. Since radian measure is based on the circle, we first review the the circle's properties.

The distance from the center of a circle to any point on the circle is called the **radius**, r. Twice this distance is the diameter. The area of a circle is given by the formula

$$A = \pi r^2 \tag{1.12}$$

where π is a transcendental number (it's an infinite nonrepeating decimal which is also not the square root of an integer). It's value is approximately $\pi = 3.14159265...$[6] The perimeter of a circle is called the **circumference**. It is a little more than six times the radius:

$$C = 2\pi r \tag{1.13}$$

These formulas could be determined by just drawing lots of circles.[7] These formulas are some of the most beautiful theorems in classical geometry.

We can use the circumference formula to determine the length of an arc. For example, a 45° circular arc has one-eighth the length of a whole circle, or $2\pi r/8 = \pi r/4$.

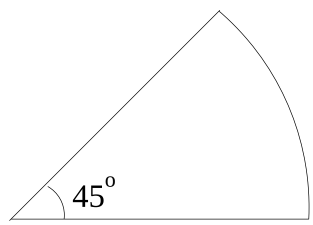


Figure 1.8: A 45° arc.

If the circle is chosen to have a radius of one, then the arc length is just $\pi/4$. This is used as an alternate way of specifying angles. **An angle in radians is represented by the length of a unit-radius arc**. For example, instead of 45°, we use $\pi/4$ radians. No wonder we use degrees instead of radians for everyday usage! However, for mathematical calculations, it is far easier to use radian measure[8]. One can relate the two:

$$\frac{Angle\ in\ radians}{Angle\ in\ degrees} = \frac{\pi}{180} \tag{1.14}$$

This conversion factor can easily be remembered since a full circle is 2π radians and is also 360°. (Thus, π radians is 180°.)

For the above example,

$$\frac{Angle\ in\ radians}{45} = \frac{\pi}{180} \tag{1.15}$$

Cross multiplying yields

$$Angle\ in\ radians = \frac{45\pi}{180} = \frac{\pi}{4} \tag{1.16}$$

Thus, 45° is $\pi/4$ radians, as obtained above.

[6] A common (but bad) fractional approximation is 22/7 which is only accurate to 2 decimal places. The approximation 355/113 is accurate to 6 decimal places.

[7] Or use a cone, a string, a calipers, and a measuring tape.

[8] In advanced mathematics it only makes sense to use radian measure. For example, it can be shown that $sin(\theta) \approx \theta$ for small angles θ, but only if we measure θ in radians

1.6 The Unit Circle

Since right triangles have only two sides which are independent, we can arbitrarily choose the length of one of the sides.[9] Let's choose the longest side to be of length 1. Notice that I didn't say one what - could be one meter, one mile, or one inch. Besides saying the length of the hypotenuse is 1, let's also specify that one of the corners of our triangle lies on some xy coordinate system (in the figure below, the solid dots represents the origin). Furthermore, the corner must one of the two corners that touches the hypotenuse. The angles are always measured from the x-axis. Let's look at some "triangles" as shown in Fig. 1.9.

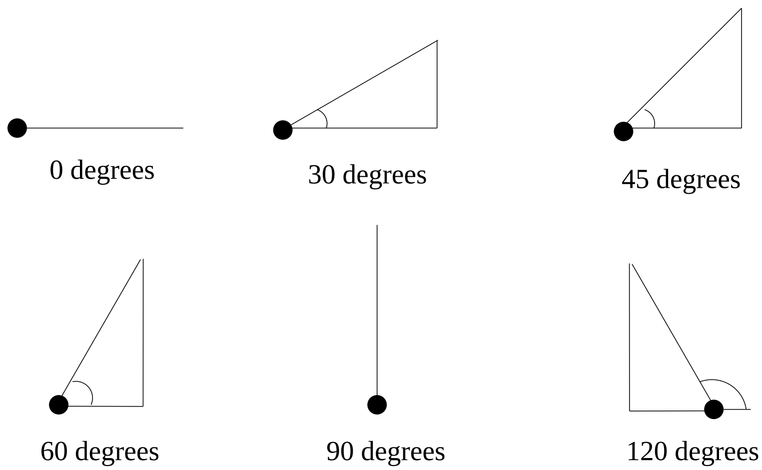


Figure 1.9: Six Triangles with a common origin.

The first triangle in the figure is at 0°. It's not much of a triangle. Since the longest side is 1, the base (x-coordinate) is also 1. The height (y-coordinate) of the triangle is 0 ($1^2 + 0^2 = 1^2$). The second triangle has a 30° angle. It is not obvious what the length of the sides are. This triangle is discussed later[10]. The third triangle is at 45°. It has the same base as height. If we call this value x, then from the Pythagorean Theorem

$$x^2 + x^2 = 1^2 \quad \Rightarrow \quad 2(x^2) = 1 \quad \Rightarrow \quad x^2 = 1/2 \quad \Rightarrow \quad x = 1/\sqrt{2} = \sqrt{2}/2 \tag{1.17}$$

The 60° triangle is the same as the 30° triangle with the base and height exchanged. The fifth triangle has zero base, but a height of 1. It has a 90° angle. The last triangle in the figure has an angle of 135°. The base is $-\sqrt{2}/2$. (Yes, a negative value!)[11] The height of the triangle is a positive $\sqrt{2}/2$.

A couple of quick notes before we look at the trig functions. Right triangles with angles between 0° and 90° are said to be in the "first quadrant." Triangles with angles between 90° and 180° are said to be in the "second quadrant." Triangles with angles between 180° and 270° are said to be in the "third quadrant." Triangles with angles between 270° and 360° are said to be in the "fourth quadrant." A triangle with an angle of 405° is the same as a triangle with an angle of 45°, so that we can add or subtract 360°'s as often as we wish.[12] For example, a triangle with an angle of −30° is the same as a triangle with an angle of 330°, and thus is in quadrant IV. Again, get use to the idea of negative sides and angles, which we must have when using an xy coordinate system.

[9] As long as we don't choose zero, of course!

[10] For those who can't wait, the short side is 1/2 and the longer side is $\sqrt{3}/2$.

[11] Negative angles may take a little bit of getting used to, but wait, it gets weirder!

[12] I told you it gets weirder. You can tell a friend you climbed stairs oriented at 405°! It makes more sense if you think about a car in a spin-out. A 45° spin-out is very different from a 405° spin-out! But, if we're only interested in the final orientation of the car, then the results are the same.

Now, let's look at the *sin* and *cos* trig functions. All of the other trig functions can be obtained from those two. The "adjacent" side is always on the x-axis, and the "opposite" side is always on the y-axis. Since the hypotenuse is always 1,

$$sin(\theta) = opposite = height = y - axis \tag{1.18}$$

and

$$cos(\theta) = adjacent = base = x - axis \tag{1.19}$$

where, by convention, we use the Greek letter θ (called "theta") to represent the angle.

Looking at the triangles in Fig. (1.9), the reader should be able to confirm that:

θ	$sin(\theta)$	$cos(\theta)$
$0°$	0	1
$45°$	$\sqrt{2}/2$	$\sqrt{2}/2$
$90°$	1	0
$135°$	$\sqrt{2}/2$	$-\sqrt{2}/2$
$180°$		
$225°$		
$270°$		
$315°$		
$360°$		

(**Yes, actually take a minute or two and confirm these NOW! Then Fill in the rest of the table, we'll need the answers in a bit.**)

When we examined the previous figure, we said that all of the solid dots were at the origin. With this, we can represent a right triangle as a single point in the xy plane. The point used is the other end of the hypotenuse. Thus, for example, the $0°$ "triangle" would be represented by the point at x=1, y=0, or (1,0). Similarly, the $45°$ triangle would be represented by a point at ($\sqrt{2}/2$, $\sqrt{2}/2$). Since we decided to always make the hypotenuse of length one, all possible triangles must be represented as some point on a circle. This is called the "unit circle."

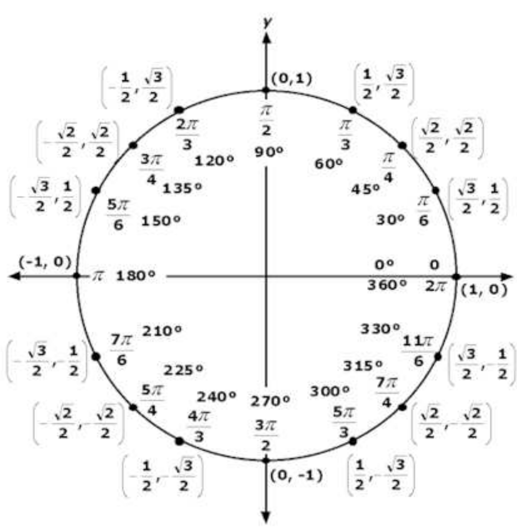


Figure 1.10: The Unit Circle.

Again, keep in mind that the x-axis can be thought of as the "cosine axis" and the y-axis can be thought of as the "sine axis." Thus, from the figure, $cos(30°) = \sqrt{3}/2$, and $sin(30°) = 1/2$.

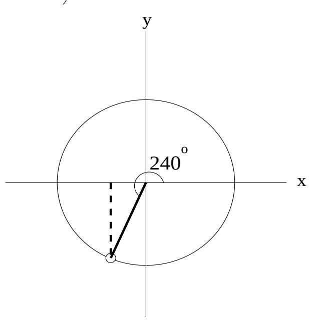
In a course in trigonometry, we are expected to have the unit circle with the values for sine and cosine of the various angles memorized! It's not really very hard. The sine and cosine values are all either 0, 1, 1/2, $\sqrt{3}/2$, or $\sqrt{2}/2$. Then you have to determine the sign. Here's how to proceed. Mentally, or actually draw a circle, and a line from the center to the point corresponding to the angle given. If the angle is 0°, 90°, 180°, or 270°, it's a 0 or a 1. Then determine the sign. If it's not one of these the answers not a 0 or a 1. These four angles are symmetric. Four other symmetric angles are 45°, 135°, 225°, and 315°. If the angle is any of these, the sine and the cosine are both $\sqrt{2}/2$. Then check for sign. Again, remember the x-axis is the cosine axis and the y-axis is the sine axis. You can visualize these in your head (we'll do an example in a moment) without the need for writing anything down. What about the others? Easy as well. There are only two possibilities - the short side is 1/2 and the long side is $\sqrt{3}/2$. Let's do an example.

Example 1.1:

Determine $sin(240°)$.

Solution 1.1:

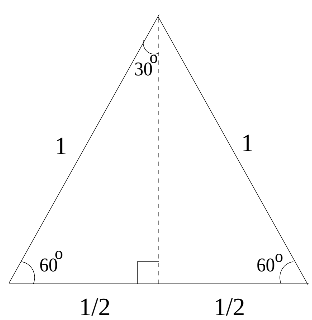
First, draw a circle and a line corresponding to 240°*:*



Since the sine axis is the y-axis, $sin(240°)$ *is the length of the dotted line in the figure. It's not one of those evenly space angles, so the answer is either* $1/2$ *or* $\sqrt{3}/2$*. It's the long one, so it must be* $\sqrt{3}/2$*. Checking for sign, we can seen that the y-coordinate value of the point is negative, so* $sin(240°) = -\sqrt{3}/2$.

Checking with the calculator (in degrees mode, of course), $sin(240°) = -.866025$*. However,* $-\sqrt{3}/2 = -.866025$ *in agreement with our calculation.* ♣

Let's go back and figure out the lengths of the sides of a 30°/60° triangle. We start by drawing an equilateral triangle (all sides are have length 1). Since all the angles are the same and the sum of the angles must be 180°, they must be 60° each.

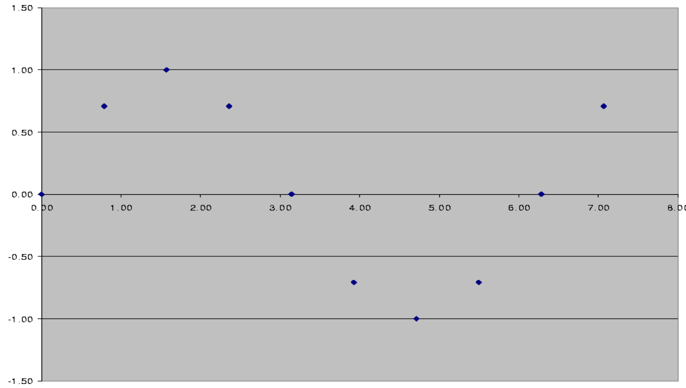


It can easily be seen that by cutting one of the 60° angles in two, 2 of the sides of the resulting right triangle are 1 and 1/2. The length of the third side, from the Pythagorean theorem is

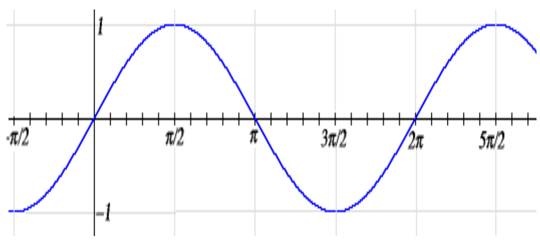
$$b = \sqrt{c^2 - a^2} = \sqrt{3/4} = \sqrt{3}/2 \tag{1.20}$$

1.7 Visualizing the Trigonometric Functions

It is often useful to have a graph of the trigonometric functions in your mind so that you can do simple sense checks for your calculations. Using the table of the sine and cosine function that you (dutifully) filled in, we can make an initial plot of the sine function:

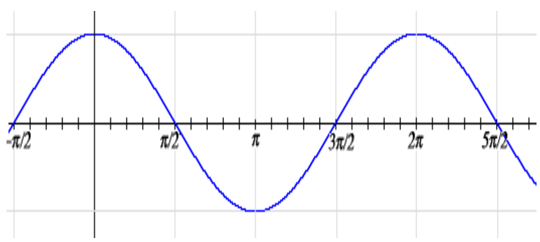


Again, to fill in the data points all we need to is to obtain a protractor to help you draw triangles with various known angles. Measure each triangle's base height and hypotenuse. The sine is just the height divided by the hypotenuse. Record the points in a table, fill in the graph. If you were to do this, your plot would look something like

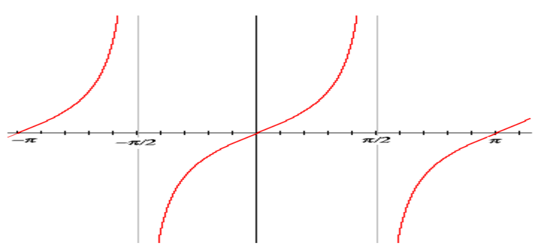


The function has a maximum value of 1, a minimum value of -1, and an average value of zero. It is periodic with period 2π.

The next plot is of cos(x). It has the same maximum, minimum, average, and period as sin(x). Indeed, it is merely a sine wave shifted a quarter of a cycle.



The following is a plot of tan(x). It has a period of π and has a vertical asymptote at $\pi/2$ and every π thereafter.



1.8 Some Applications

Trigonometry underlies just about everything remotely connected to "high-tech." Trigonometry is used in everything from building bridges to finding stars to surveying. We could come up with literally thousands of examples, and it would still fall woefully short of the power of trigonometry. We'll start with an example from optics.

Application 1.3:

Different materials have different optical properties. One parameter that demonstrates this is refractive index. For example, air has a refractive index of 1.0, water is 1.33, and most glass is about 1.5. When light strikes a boundary between two different media, some of the light is reflected off and some is transmitted through (or, "refracted"). The light that is reflected bounces off at the same angle it hits with much like a billiard ball hitting a pool table cushion (this is the "law of reflection"). The light that is refracted does not go straight through - it goes at an angle given by Snell's Law:

$$n_1 sin(\theta_1) = n_2 sin(\theta_2) \tag{1.21}$$

If sunlight hits a lake at 37° *(from the normal), what angle does it travel through the water?*

In this case, Snell's Law is

$$n_{water} sin(\theta_{water}) = n_{air} sin(\theta_{air}) \tag{1.22}$$

or

$$1.33 sin(\theta_{water}) = sin(37^\circ) \tag{1.23}$$

Dividing both sides by 1.33 and taking the inverse sine yields $\theta_{water} = 26.9^\circ$ *(from the normal).*

While this is a trivial example, many scientists have spend their entire lifetimes designing telescopes and optical fiber systems basically using only these two laws. Their field is known as "Geometric Optics." ♣

Application 1.4:

You see an airplane traveling in the sky at 18° *from the horizontal. One minute later it is at* 30° *from the horizontal. Assuming that the airplane is traveling at about 550 miles/hour, what is the cruising altitude of the airplane?*

We begin by drawing a picture:

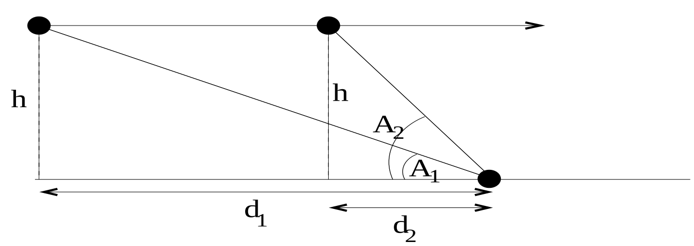


Figure 1.11: Determining the cruising altitude of an airplane.

Since the plane is traveling at 550 miles/hour for 1/60 of an hour, we know that the distance it travels is $550/60 = 9.167$ *miles. With respect to the figure, this is* $d_1 - d_2$*. We also know* A_2 *and* A_1*. The obvious solution is to look at the triangle contained by the three large dots. However, we* *only know one side and one angle of this triangle, so the "obvious solution" doesn't work! Looking at the two right triangles in the figure, it follows that*

$$\frac{h}{d_1} = tan(\theta_1) \Rightarrow d_1 = \frac{h}{tan(\theta_1)} \tag{1.24}$$

$$\frac{h}{d_2} = tan(\theta_2) \Rightarrow d_2 = \frac{h}{tan(\theta_2)} \tag{1.25}$$

Again, we don't know either d_1 or d_2, but $d_1 - d_2$ is known. Solving these for $d_1 - d_2$ yields

$$d_1 - d_2 = \frac{h}{tan(\theta_1)} - \frac{h}{tan(\theta_2)} = h[cot(\theta_1) - cot(\theta_2)] \tag{1.26}$$

and h is the only unknown. Solving for h yields

$$h = \frac{d_1 - d_2}{cot(\theta_1) - cot(\theta_2)} \tag{1.27}$$

Plugging the given data into our derived formula yields

$$h = \frac{9.167}{cot(18^\circ) - cot(30^\circ)} \tag{1.28}$$

Many calculators do not have the cotangent function, so we must rewrite the solution in terms of the tangent function:

$$h = \frac{9.167}{1/tan(18^\circ) - 1/tan(30^\circ)} \tag{1.29}$$

Of course we must be sure our calculator is in "degrees mode" and not "radian mode." If the calculator gives no indication, it is in degrees mode. The result of the calculation is 7.29 miles. ♣

1.9 Trigonometric Identities

There are a number of mathematical identities relating the trig functions to each other. In this section we will examine some of these.

A restatement of the Pythagorean Theorem in terms of trig functions is

$$sin^2(\theta) + cos^2(\theta) = 1 \tag{1.30}$$

which is obtained by applying the definitions of sine Eq. (1.4) and cosine Eq. (1.5) into Eq. (1.2).

As discussed, the trig functions are the same for 45° as 405°. We can add or subtract any (integer) number of 360°'s (or 2π's). Put another way, the sine and cosine functions are periodic with period 2π, so that

$$sin(\theta \pm 2\pi) = sin(\theta) \tag{1.31}$$

$$cos(\theta \pm 2\pi) = cos(\theta) \tag{1.32}$$

$$tan(\theta \pm \pi) = tan(\theta) \tag{1.33}$$

Similarly, it can be seen from the graphs that

$$sin(\theta \pm \pi) = -sin(\theta) \tag{1.34}$$

$$cos(\theta \pm \pi) = -cos(\theta) \tag{1.35}$$

$$tan(\theta \pm \pi/2) = -cot(\theta) \tag{1.36}$$

Another thing to notice from the plots of the trigonometric functions is that the sine function is merely a cosine function phase shifted (and vice-versa) so that

$$sin(\theta \pm \pi/2) = \pm cos(\theta) \tag{1.37}$$

$$cos(\theta \pm \pi/2) = \mp sin(\theta) \tag{1.38}$$

Looking at the graphs of the trigonometric functions, it can be seen that sine is an odd function (anti-symmetric about the y-axis) and cosine is an even function (symmetric about the y-axis) so that

$$sin(-\theta) = -sin(\theta) \tag{1.39}$$

$$cos(-\theta) = cos(\theta) \tag{1.40}$$

$$tan(-\theta) = -tan(\theta) \tag{1.41}$$

The previous identities merely involve understanding how sine and cosine work. Therefore, most of them do not require memorization as long as you remember what the plots of sine and cosine look like. Some less obvious, but extremely powerful identities are the identities for a sum of angles:

$$cos(A \pm B) = cos(A)cos(B) \mp sin(A)sin(B) \tag{1.42}$$

$$sin(A \pm B) = sin(A)cos(B) \pm cos(A)sin(B) \tag{1.43}$$

$$tan(A \pm B) = \frac{tan(A) \pm tan(B)}{1 \mp tan(A)tan(B)} \tag{1.44}$$

You should note that the previous identities (except the first one) can be obtained from these last three "sum" identities. There are many other trigonometric identities, but these are the basic ones.

We now have a good understanding of right triangles, and we've looked at some applications in physics. As an example of geometry, we apply these results to a higher-order figure - the non-right triangle.

1.10 Non-right Triangles

We could build an arbitrary non-right triangle out of two triangles and proceed to determine relationships between the non-right triangles angles and sides. If we were to do this we would obtain the "law of sines" and the "law of cosines:"

law of sines	$\frac{a}{sin(A)} = \frac{b}{sin(B)} = \frac{c}{sin(C)}$
law of cosines	$c^2 = a^2 + b^2 - 2ab\cos(C)$

where the length of the side a is on the opposite side of the angle A. Same applies to the length b and the angle B, and the length c and the angle C. Furthermore, we could obtain the formula for the area of a non-right triangle:

$$Area = \sqrt{s(s-a)(s-b)(s-c)} \tag{1.45}$$

where

$$s = \frac{a+b+c}{2} \tag{1.46}$$

This area formula was first obtained by Hero of Alexandria (65-125 A.D.), and is sometimes called "Heron's Formula."[13]

Disregarding area and perimeter for a moment, a triangle has six parameters - three sides and three angles. The basic result of the sine and cosine laws is that given any three (independent) pieces, we can

[13]He is known both as Hero and Heron.

determine the other three.[14] In some cases, if two angles and a single side is given, there are two possible triangles, and these laws can be used to find the parameters of both.

Example 1.2:

Determine everything about the following triangle:

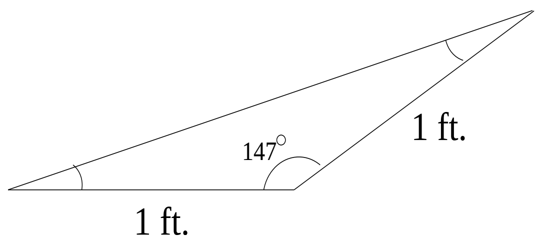


Figure 1.12: SAS - side/angle/side are given.

Solution 1.2:

Since we do not have an angle with a corresponding opposite side given, we cannot use the law of sines. From the law of cosines it follows that the length of the hypotenuse can be obtained from

$$c^2 = 1^2 + 1^2 - 2\cos(147^\circ) = 3.67734\ ft^2 \tag{1.47}$$

Square rooting both sides yields

$$c = 1.9176\ ft \tag{1.48}$$

After making calculations of this sort, one MUST examine the answer for reasonableness (a reality check). In this case, we know that the answer less than 2 (if given angle was 180° *instead of* 147°*) and greater than* $\sqrt{2} \approx 1.414$ *(if given angle was* 90°*). The answer of* 1.9176 *is indeed between these two values.*

From symmetry, both the other angles are the same. Since the sum of the angles of a triangle must add up to 180°*, the unknown angles are each*

$$\frac{180^\circ - 147^\circ}{2} = 16.5^\circ \tag{1.49}$$

Without the insight of symmetry, we could use the law of sines:

$$\frac{sin(\theta)}{1} = \frac{sin(147^\circ)}{1.9176} = 0.284021 \tag{1.50}$$

Taking the inverse sine of both sides yields

$$\theta = sin^{-1}(0.284021) = 16.5^\circ \tag{1.51}$$

which is the same as calculated above.

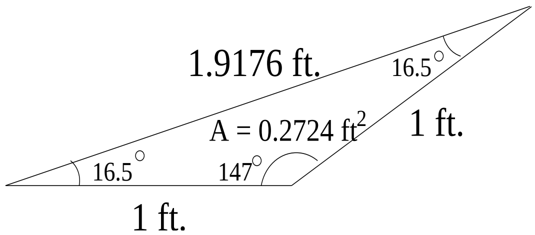
Finally, the area of the triangle can be found by first finding s*:*

$$s = \frac{1 + 1 + 1.9176}{2} = 1.9588 \tag{1.52}$$

[14]We can't find the lengths of the sides if only the three angles are given (because we could double the size of the triangle and the angles would be the same). The only other thing we can tell is if the person telling us is completely full of baloney. They are if the angles don't add up to 180°!

Then, from Heron's Formula, the area is[15]

$$A = \sqrt{(1.9588)(0.9588)(0.9588)(1.9588 - 1.9176)} = 0.2724\ ft^2 \tag{1.53}$$



♣

1.11 Closing Notes

1.11.1 Summary

Trigonometry is computational geometry. Since complicated figures can be broken into right triangles, results learned about right triangles can be applied to these figures. This is why one would want to study trigonometry (we want to study complicated figures). However, this is only half of the reason. Trigonometry can be applied to a variety of physical situations where there are no figures or triangles present. Recall the problem of finding the height of a pyramid. We created imaginary triangles that only existed in our minds to solve the problem. So, if we want to study the physical world, we are motivated to study trigonometry.

We started by examining the relationships between the lengths of the sides and angles of right triangles. How did we do it? We used two steps. First, we noticed that when we draw two sides of a triangle, we have no choice as to the length of the third side. Therefore, there must be some relationship between the sides of a right triangle. So, now that we know that such a relationship exists, we merely need to draw lots of triangles, make a table of the lengths of sides, and notice a pattern. Now, this may take a little time (somewhere between half an hour to ten hours, but NOT a lifetime or anything), but it is certainly a reasonable way to proceed. When we are finished we will have "discovered" the Pythagorean Theorem.

Now, we want to determine a relationship between the angles of a right triangle. What do we do? Same thing - draw some triangles. It should be clear very quickly that the interior angles of a right triangle (or for any triangle for that matter) add up to 180°[16].

Thus, given the lengths of any two sides of a right triangle, we can determine the length of the remaining side. Given one of the angles of a right triangle we can determine the other (the third angle is always 90°). What if we are given one angle and one side? I hope you agree that the sensible thing to do is to look at many (different sized) triangles with the same angles - these are similar triangles. Drawing a few and making a table with the lengths of the sides, it immediately can be seen that the ratio of any two sides is constant - this is the law of similar triangles.

Next, we give these ratios names. For example, the ratio of the opposite side to the adjacent side is the tangent function. Again, we draw triangles with different angles and make a table of different angles vs. the tangent function. We do the same for the other trigonometric functions. We then make many copies of this so it doesn't get lost. We keep these tables with us whenever we want to calculate anything. This keeping of "sacred" tables is a time honored tradition in mathematics that still exists today. For example, you may have a copy of these trigonometric tables with you now. They are stored in your calculator.

A right triangle has three sides and two independent angles, or five pieces of information. Using the Pythagorean theorem, the sum of angles being 180° rule, and the trigonometric functions, we can determine all five given any two. We can apply this ability to an absolutely insane number of problems in geometry

[15]Following the same logic in the previous reality check, you should be able to show that the area must be between 0 ft^2 and 0.5 ft^2.

[16]The proof is not very hard. I thought it up in about 10 seconds. Just bisect an isosceles triangle.

and physics. As examples, we looked at non-right triangles (geometry) and the calculation of the height of a pyramid (physics).

1.11.2 What more could we do?

We've used **inverse trig functions** in the examples, but haven't talked about them much. One thing to keep in mind is you have to be careful sometimes since their graphs are not single-valued. What's the inverse cosine of 3? Well, friend there's an infinite number of answers, which one do you want? If this bothers you, don't worry too much, just use the answer your calculator tells you. If we were to continue studying trig, the very next thing to do would be to study the inverse trig functions.

Vectors go with trigonometry like peas and carrots, though sometimes vectors can be used instead of trigonometry. For example, it's relatively easy to derive the law of sines and law of cosines using vectors instead of drawing triangles. As another example, consider this problem: Joe walks 2 miles north by northeast (picks up some lunch), then 1 mile southwest (delivers some mail), and finally 1.5 miles south. Where should you walk if you want to meet up with Joe? You can do it by drawing lots of triangles and applying trig laws, but it's much easier using vectors. Therefore, another important area we could explore is vector algebra (and later vector calculus!).

We've already hinted at the use of polar coordinates, but other **coordinate systems** are also used extensively in mathematics and physics. Conversion between these coordinate systems involves heavy use of trig identities. As an example, if we wanted to specify where a particular star was right now, we could specify an azimuth angle (facing north, how many degrees do we have to turn to face the star), an angle of elevation (what angle do we tilt our heads at to see the star), and the distance to the star. This alternate method of specifying the location of the star is called spherical coordinates. One encounters coordinate systems in physics early and often, we could also study this.

Most of the trig identities can be derived using the Euler relation - a theorem that uses **complex numbers** to relate exponentials to the trig functions. Complex numbers are often left out of many curricula, but you can't do much AC circuit analysis without it.

The trigonometric functions can all be represented by a **"Taylor Series"** - an infinite series of polynomials (For example, $sin(x) = x - x^3/3! + x^5/5! + ...$). One of the things you should have learned is that all problems cannot be solved in closed form. In physics, we don't care, we still need to know the answer. Taylor series help us out.

An arbitrary periodic function can be represented as an infinite sum of sine and cosine function as a **"Fourier Series"**. If you want to look at data, you should want to look at Fourier Series. My personal opinion is that every biologist (who collects data) should be exposed to it.

1.11.3 We've conquered trigonometry, now what?

If trigonometry is generally regarded as daunting, calculus is thought of as unlearnable and uninteresting. Actually, calculus is simple and profound. Basically, calculus comes in two flavors, differential calculus and integral calculus. Differential calculus comes from studying slopes of curves and integral calculus comes from examining areas under curves.

Think about what the speedometer on your car is telling you. Draw some velocity vs time graphs. Based on your graphs think about how to draw corresponding position vs. time graphs and acceleration vs. time graphs. If you take this seriously at all, you'll invent calculus *whether you want to or not.* The same types of steps we've used here to reason out how trigonometry can be used for calculus as well, but that's another story...

Here is the answer to the Pythagorean Triplet problem.[17]

[17] The next triplet is $14^2 + 48^2 = 50^2$. This follows since if $a^2 + b^2 = c^2$, then (multiply both sides by 4 yields) $(2a)^2 + (2b)^2 = (2c)^2$. The 14 triplet is just double the 7 triplet.