Complex Numbers Review and Tutorial

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Tutorial 1

Complex Numbers

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1.1 Introduction

A complex number is a number of the form

$$z = a + ib \tag{1.1}$$

where the imaginary unit is defined as

$$i = \sqrt{-1} \tag{1.2}$$

and a is the real part of z, written $a = Re(z)$, and b is the imaginary part of z, written $b = Im(z)$. First, note that a and b are real numbers, only z is a complex number.[1]

The convention used here is that the real part is always written before the imaginary part, and that the imaginary unit, i, is written before the imaginary part, as in Eq. (1.1). In electrical engineering, a lower case i represents a time-dependent current, so it is their convention to use the symbol j as the imaginary unit.

It may be easily verified from Eq. (1.2) that $i^2 = -1$, and that

$$\frac{1}{i} = -i \tag{1.3}$$

1.2 The Complex Plane

A complex number can be visualized in a two-dimensional number line, known as an Argand diagram, or the complex plane as shown in Fig. 1.1. The complex plane replaces the number line as a visualization tool

[1] **Do NOT** use the language "Imaginary Numbers". Complex numbers have real and imaginary *parts*. However, if a number has no real part, then it is called 'pure imaginary'.

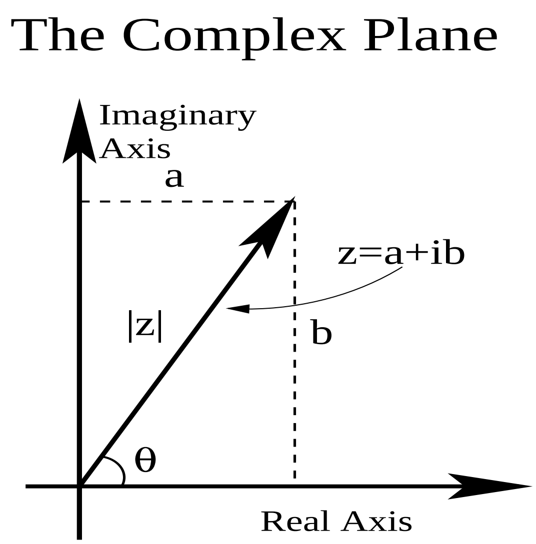

Figure 1.1: A complex number is easily visualized as a ”phasor” in the complex plane.

for real numbers. However, rather than plot points in the complex plane, it is conventional to represent a complex number as a vector in the complex plane. Instead of calling them complexified vectors, they are referred to as “phasors.” Note that because the visualization is 2-dimensional, a polar form for complex numbers is suggested. This is discussed below.

As an addition side note about the complex plane, complex analysis can be used to study improper real integrals that couldn't otherwise be solved. Basically, for an integral that extends from $-\infty$ to ∞, instead of integrating over the real number line, one may perform a closed path line integral that includes the real number line and a semicircle of radius R, which is allowed to approach ∞. The residue theorem relates the closed loop integral to a real value. However, it can be shown that the integral over the infinite semicircle approaches zero, and our integral of interest is obtained.

1.3 Complex Magnitude

From Fig. 1.1, it can be easily seen (using the Pythagorean theorem) that the magnitude, or length, of the vector representing a complex number is

$$|z| = \sqrt{a^2 + b^2} \tag{1.4}$$

Thus, the complex magnitude is the square root of the sum of the squares of the real and imaginary parts of the complex number. This definition generalizes the absolute value function of a real number.

1.4 Complex Conjugate

The complex conjugate of a complex number z, is denoted z^*, and is defined

$$z^* = a - ib \tag{1.5}$$

Sometimes it is useful to think of the conjugation process as “replacing i with $-i$.” Where would the vector z^* fit on the complex plane in Fig 1.1?

Example 1.1:

Complex Conjugate
Prove that $z^* z = |z|^2$.

Solution 1.1:

$$\begin{aligned} z^*z &= (a-ib)(a+ib) \\ &= a^2+b^2 \\ &= (a^2+b^2)^{1/2}(a^2+b^2)^{1/2} \\ &= |z|^2 \end{aligned}$$

♣

Note that z^*z is real and positive. A quotient of complex numbers can be written separated into real and imaginary parts using the above conjugate relation as shown in the next example.

Example 1.2:

Complex Fractions

Separate the complex number $z = \frac{3-i4}{5+i12}$ into real and imaginary parts.

Solution 1.2:

The solution involves the well known rationalizing the denominator technique of multiplying the top and bottom of the quotient by the conjugate of the denominator.

$$\begin{aligned} z &= \frac{3-i4}{5+i12} \\ &= \frac{3-i4}{5+i12}\frac{5-i12}{5-i12} \\ &= \frac{(15-48)+i(-20-36)}{13} \\ &= -\frac{33+i56}{13} \end{aligned}$$

♣

It may be noted that $|z^*| = |z|$. Also, the conjugate of a product is a product of conjugates so that $(uv)^* = u^*v^*$. Similarly, the conjugate of a sum is the sum of conjugates so that $(u+v)^* = u^* + v^*$. Finally, the conjugate of a conjugate is the function itself, i.e. $(z^*)^* = z$. Put another way, complex conjugation "toggles."

Example 1.3:

Complex Functions

Consider the function $f(z) = 3z^2 + (2+i7)z + i6$ where z is a complex variable.
a) What is the conjugate of the function, $f^*(z)$?
b) What is $f(z^*)$?
c) What is $f^*(z^*)$?

Solution 1.3:

Following the simple steps:

$$\begin{aligned} a) f^*(z) &= [3z^2 + (2+i7)z + i6]^* \\ &= (3z^2)^* + [(2+i7)z]^* + (i6)^* \\ &= 3z^{*2} + [(2+i7)^*z^*] + i^*6^* \\ &= 3z^{*2} + (2-i7)z^* - i6 \\ b) f(z^*) &= 3z^{*2} + (2+i7)z^* + i6 \\ c) f^*(z^*) &= 3z^2 + (2-i7)z - i6 \end{aligned}$$

♣

1.5 Polar Form of a Complex Number

Examining Fig. 1.1, it can be seen that a complex number can be expressed in terms of a magnitude and an angle as

$$z = |z|[cos(\theta) + isin(\theta)] \tag{1.6}$$

Though not derived here, mathematician Leonard Euler proved last century that

$$e^{i\theta} = cos(\theta) + isin(\theta) \tag{1.7}$$

This is one of the most powerful results in all of mathematics. Basically, all of the trig identities can be derived from it. Substituting the Euler Relation [2] Eq. (1.7) into Eq. (1.6) yields

$$z = Ae^{i\theta} \tag{1.8}$$

where A is defined as the magnitude of the complex number z. Thus, a complex number can be thought of as having two forms: a rectangular form (Eq. 1.1) and a polar form (Eq. 1.8). The complex conjugate of Eq. (1.7) is

$$e^{-i\theta} = cos(\theta) - isin(\theta) \tag{1.9}$$

Adding Eqs. (1.7) and (1.9) and dividing by 2 yields the important result

$$cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{1.10}$$

Similarly,

$$sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{1.11}$$

While the Euler relation is the most important result here, these two closely follow. While extremely useful, it is also generally interesting that the sum of complex functions yields a real function.

In the polar form the conjugate of z is

$$z^* = Ae^{-i\theta} \tag{1.12}$$

Using the polar form, the result from Example A.1 becomes transparent.

Example 1.4:

The Euler Relation

Use Euler's identity to derive the formula for the cos of the sum of two angles.

Solution 1.4:

If $Re\{\}$ represents an operator which takes the real part of a complex number or function, then from Eq. (1.10)

$$\begin{aligned}
cos(a+b) &= Re\{e^{i(a+b)}\} \\
&= Re\{e^{ia}e^{ib}\} \\
&= Re\{[cos(a) + isin(a)][cos(b) + isin(b)]\} \\
&= Re\{[cos(a)cos(b) - sin(a)sin(b)] + i[sin(a)cos(b) + cos(a)sin(b)]\} \\
&= cos(a)cos(b) - sin(a)sin(b)
\end{aligned}$$

Clearly the $sin(a+b)$ is also readily obtained. ♣

[2] also called Euler's Identity, as if Euler only came up with one identity!

1.6 Hyperbolic Sin and Cos

It is clear that the *sin* and *cos* of a real number is a real number, but what about the *sin* and *cos* of a number that is pure imaginary? From Eqs. (1.10) and (1.11) it follows that the *sin* and *cos* of a pure imaginary number is ... drumroll, please ... real! This was the inspiration for defining hyberbolic *cos* and *sin*. They are defined by simply erasing the "i's" in Eqs. (1.10) and (1.11):

$$cosh(\theta) = \frac{e^{\theta} + e^{-\theta}}{2} \tag{1.13}$$

$$sinh(\theta) = \frac{e^{\theta} - e^{-\theta}}{2} \tag{1.14}$$

Of course, once you have *sinh* and *cosh*, you can define *tanh*, *coth*, *arcsinh*, ... In addition, there are hyberbolic trigonmetric identities. For example, by looking at the above equations, you should be able to confirm (without pencil and paper!) that

$$cosh^2(x) - sinh^2(x) = 1 \tag{1.15}$$

The switching from the variable θ to x was intentional. The argument of a sinusoid is an angle (in radians), while the argument of a hyberbolic sinusoid is not (it's dimensionless).

1.7 Converting Between Polar and Rectangular Forms

A complex number written in polar form may be converted to rectangular form by the relations

$$a = Acos(\theta) \tag{1.16}$$

$$b = Asin(\theta) \tag{1.17}$$

These are immediately obtained by substituting the Euler relation into the polar form of a complex number. Conversely, these equations may be inverted, and a complex number written in rectangular form may be converted to polar form by the relations

$$A = \sqrt{a^2 + b^2} \tag{1.18}$$

$$\theta = tan^{-1}(b/a) \tag{1.19}$$

These four formulas are identical to normal polar-to-Cartesian and vice-versa conversions. Quite often in complex number calculations, one switches between the two forms.

1.8 Powers and Roots of Complex Numbers

A most logical way to continuing our study of complex numbers would be to look at the *sin* and *cos* of a complex number, the exponential function of a complex number, powers and roots of complex numbers... Basically, separate any elementary function[3] of a complex number into real and imaginary parts. The *sin*, *cos*, and exponential functions are easy, and left to the reader as an exercise. Here we consider powers and roots of complex numbers.

As a first step in this method is to write your complex number in polar form. With this done, the power of a complex number is easily calculated:

$$z^n = A^n e^{in\theta} \tag{1.20}$$

If desired (or required), one would then convert this back to the rectangular form. Solving problems involving complex numbers and functions often involves switching back and forth between rectangular and polar form.

Roots of complex numbers may be obtained in a nearly identical manner:

$$z^{1/n} = A^{1/n} e^{i\theta/n} \tag{1.21}$$

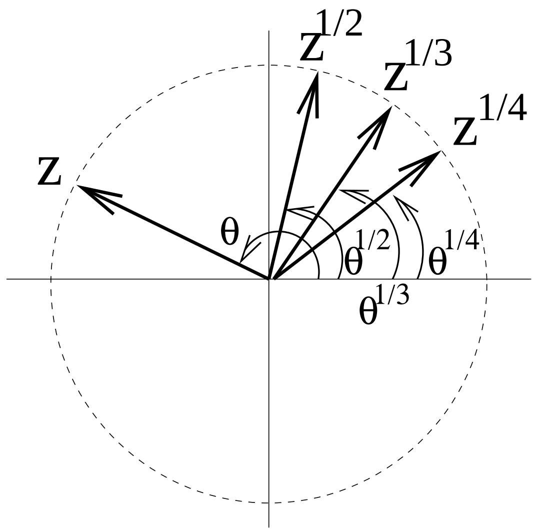

Figure 1.2: The nth root of a complex number is an angle which $1/n$th the original number.

It is interested and inciteful to interpret these results graphically. The angle is reduced by a factor of $1/n$ and the magnitude is affected in the same way as the square root of a real number is. For a complex number of unit magnitude, a plot of a complex number with three of its roots are shown. At times it is useful to have the formula for a root of a complex number in rectangular form. While this can't be done in general, the square root is tractable:

$$\sqrt{a+ib} = \sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + isgn(b)\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}} \tag{1.22}$$

where $sgn(b)$ is the signum function, which is also known as the sign of b. The signum function is generally defined as $b/|b|$. We define $sgn(0)$ to be unity.

1.9 The Problem that "Can't Be Done"

In pre-calculus and even in calculus, you may have been told that calculating the *arccos* of a number greater than 1 can't be done. Since the *cos* function oscillates between -1 and 1, calculating $arccos(3)$ would be "difficult." When I ask my calculator to calculate $arccos(3)$, it says "Error 0."

Of course the calculator and the early math courses are restricting themselves to the real number system.[4] The $arccos(3)$, for example, is just a complex number. In fact, it is pure imaginary, as we will now show.

Example 1.5:

Inverse Trig Functions
Calculate $arccos(3)$.

Solution 1.5:

let $y \equiv arccos(3)$, *then* $cos(y) = 3$. *But from the Euler Relation, the cos function can be written as a sum of complex exponentials:*

$$\frac{e^{iy}+e^{-iy}}{2} = 3 \tag{1.23}$$

[3]and special functions such as Bessel functions too!
[4]Your calculator may not have this restriction.

Multiplying both sides by $2e^{iy}$, and moving all terms to the left hand side of the equation results in

$$\left(e^{iy}\right)^2 - 6e^{iy} + 1 = 0 \tag{1.24}$$

which is a quadratic equation in e^{iy}. The solutions are

$$e^{iy} = 3 \pm 2\sqrt{2} \tag{1.25}$$

Taking the natural log of both sides and multiplying by $-i$ results in

$$y = -iln(3 \pm 2\sqrt{2}) \tag{1.26}$$

♣

1.10 PROBLEMS

P1.1. Separate the following into real and imaginary parts:
a) $\frac{3+i4}{5+i7}$
b) $(3+i4) + i(4+i5) + (2+i3)(4+i5)^2$
c) $tan(3+i4)$
d) e^{3+i4}
e) $\sqrt{1+i2}$
f) $ln(3+i4)$
g) $sin^{-1}(3)$
h) i^i
Hint: i can be thought of as a complex number in rectangular form.
i) There are an infinite number of values for i^i, what are they?
Hint: $e^{i\theta} = e^{i(\theta+2n\pi)}$

P1.2. Consider a series AC electrical circuit with two resistors and a capacitor. The output complex voltage

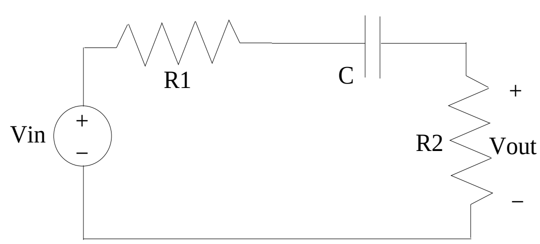

Figure 1.3: A simple AC circuit.

is related to the input complex voltage by the voltage divider law

$$\hat{V}_{out} = \frac{R_2}{R_1 + R_2 - i/(\omega C)}\hat{V}_{in}$$

If $R_1 = 100\Omega$, $R_2 = 200\Omega$, $C = 50\mu F$, and $\omega = 2\pi(60)$ cycles/s, and $\hat{V}_{in} = 100V$, then what is the

a) magnitude of the output voltage,
b) phase of the output voltage.
c) plot $|\hat{V}_{out}/\hat{V}_{in}|$ as a function of differents ω's.

P1.3. If Ψ is the wave function from the Schrödinger equation, then the probability density of finding a particle at a particular place is given by $P(x) = \Psi^*\Psi$. Suppose that you have solved the Schrödinger equation for a given potential functions, and you find that

$$\Psi(x) = \frac{p}{1 + qx^2}$$

where $p = p_r + ip_i$, $q = q_r + iq_i$ are complex constants, and $P = p^*p$, and $Q = q^*q$. Compute the probability density distribution in this case. (Note: x is a real variable).

P1.4. One problem of interest is the writing of the n^{th} power of cos into a Fourier series. Using trigonometric identities it is generally difficult to prove that

$$cos^m(\omega t) = \frac{1}{2^m}\sum_{k=0}^{m} \frac{m!}{k!(m-k)!} cos[(m-2k)\omega t]$$

Use the Euler relation to derive the above result[5].

P1.5. Plot the following phasors tail-to-tip on a piece of graph paper:

$$2 - \sqrt{29}e^{-itan^{-1}(5/2)} + (\sqrt{34}/5)e^{+itan^{-1}(3/5)} - 2 + (\sqrt{136}/5)e^{-itan^{-1}(3/5)}$$

$$-\sqrt{5}e^{-itan^{-1}(2)} - \sqrt{5}e^{+itan^{-1}(2)} + (\sqrt{34}/5)e^{+itan^{-1}(3/5)} - (\sqrt{29}/5)e^{+itan^{-1}(5/2)} + 2$$

Hints: 1) Start near the bottom middle of your graph paper. 2) The sum of the complex numbers is zero, so the last phasor should end at the same place your first complex number started.

P1.6. The complex propagation constant for an electromagnetic wave propagating in a conductive medium can be obtained from the formula

$$\begin{aligned} {k_0}^2 &= \omega^2\mu\epsilon - i\mu\sigma\omega \\ &\equiv (\beta_0 - i\alpha_0)^2 \end{aligned}$$

where β_0 is the propagation constant and α_0 is the loss per length.

a) If the skin depth $\delta = 1/\alpha_0$, then obtain an analytical expression for δ in terms of ω, μ, ϵ, and σ.

b) Simply you expression for $\frac{\sigma}{\omega\epsilon} << 1$.

P1.7. Use the Euler Relation to derive the trigonometric identity for the sin of the sum of two different angles: $sin(a + b)$.

P1.8. Often times books show a proof of Euler's Identity by looking at the Taylor series expansion for $sin(x)$ and $cos(x)$, comparing it to the expansion for e^x and saying "Tada, it works!" Here the goal is to *derive* the Euler relation, assuming that we know a little bit about differential equations. Put another way, we don't want to just show that it happens to work, but that it *must* work. First, we consider the following differential equation which represents simple harmonic motion such as from a pendulum (small angles) or a spring:

$$\frac{d^2y}{dt^2} + \omega^2 y(t) = 0$$

Complete the following steps of the derivation:

[5] As the student may be aware, the binomial theorem is $(a + b)^m = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} a^k b^{m-k}$

a) Show that $y(t) = asin(\omega t) + bcos(\omega t)$ are solutions to the differential equation.

b) Apply the boundary conditions $y(0) = y_0$ and $y'(0) = y'_0$ to this solution.

c) Show that $y(t) = Ae^{i\omega t} + Be^{-i\omega t}$ are solutions to the differential equation.

d) Apply the boundary conditions $y(0) = y_0$ and $y'(0) = y'_0$ to this solution.

e) Since the differential equation is second order, it has only two independent solutions. Thus coefficients of the y_0 terms in the two equations must be equal to each other. Set them equal to each other, and solve for $e^{i\omega t}$.

P1.9. Make separate plots of the following roots as phasors in the complex plane:
a) $z^2 = 1$
b) $z^3 = 1$
c) $z^4 = 1$
d) $z^5 = 1$
e) $z^3 = \frac{1+i}{\sqrt{2}}$