Differential and Integral Calculus Review and Tutorial

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Tutorial 1

Calculus Review

Contents

This tutorial is a review of the basic results of differentiation and integration. Of course some of the results may be new to some of the readers. Hopefully, those readers will find the new results interesting as well as informative.

There are many things one could say about the history of calculus, but one of the most interesting is that integral calculus was first developed by Archimedes of Syracuse OVER 2250 YEARS AGO! He was a very interesting guy. You can google him to learn more, but I highly recommend the (historical fiction) book "The Sand Reckoner" by Gillian Bradshaw which is a story of his life.

1.1 What are Elementary Functions?

The Elementary functions are:

1. Polynomials (of integer and complex order)
2. Exponential and Logarmithmic Functions
3. Sinusoidal and Inverse Sinusoidal Functions
4. Hyberbolic Sinusoidal and Inverse Hyperbolic Sinusoidal Functions
5. Any finite number of Sums, Products, or Compositions of Elementary Functions

Here are some examples of elementary functions:

Elementary Function	Examples
Polynomials	$a_3x^3 + a_2x^2 + a_1x + a_0$
Exponential and Logarmithmic Functions	e^{ax}, $ln(ax)$
Sinusoidal and Inverse Sinusoidal Functions	$cos(ax)$, $tan^{-1}(ax)$
Hyberbolic Sinusoidal and Inverse Hyperbolic Sinusoidal Functions	$cosh(ax)$, $tanh^{-1}(ax)$
Composite Elementary Function	$\frac{e^{sin(x)+x^2}}{cosh(x)} + ln(7x)$

So, what isn't an elementary function? There are certain integrals and differential equations that "can't be solved" so instead of solving them, we name them. For example, Bessel functions are solutions to "Bessel's Equation." Nonelementary functions that have a special name are known as **Special Functions**. Another common example of a special function is the Error Function which is the solution to the integral

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\hat{x}^2} d\hat{x} \tag{1.1}$$

1.2 Differential Calculus

Fast Facts:

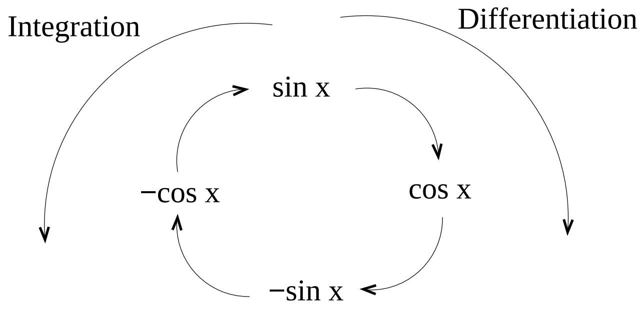
1. Definition: $f'(x) = lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.
2. Derivative is an operator (it operates on functions).
3. In particular, the derviative is the slope operator. Thus, it represents a "rate of change." When the independent variable is time, the derivative becomes a time rate of change. The time rate of change of position is velocity, the time rate of change of velocity is acceleration, and the time rate of change of acceleration is jerk. It can be readily seen that the units of the time derivative of f are $f/time$.
4. The inverse operator is the antiderivative or integral (This is the Fundamental Theorem of Calculus).
5. The Integral is the Area Operator.
6. The Derivative of any Elementary Function is an Elementary Function.

1.2.1 Some Derivatives

Here is a very short table of derivatives:

Function	Derivative
x^n	nx^{n-1}
e^x	e^x
$ln(x)$	$\frac{1}{x}$
$sin(x)$	$cos(x)$
$cos(x)$	$-sin(x)$
$-sin(x)$	$-cos(x)$
$-cos(x)$	$sin(x)$

One may notice that the derivatives for *sin* and *cos* follow a simple pattern (and that this pattern reminds one of the unit circle):



1.2.2 Basic Theorems

Theorem Name	Theorem
Chain Rule	$\frac{d}{dx}\left(A(B(x)\right) = \frac{dA(B)}{dB}\frac{dB(x)}{dx}$
Linearity	$\frac{d}{dx}\left(aA(x)+bB(x)\right) = a\frac{dA}{dx}+b\frac{dB}{dx}$
Product Rule	$\frac{d}{dx}\left(A(x)B(x)\right) = A(x)\frac{dB}{dx}+\frac{dA}{dx}B(x)$
Quotient Rule	$\frac{d}{dx}\left(\frac{A(x)}{B(x)}\right) = \frac{B(x)\frac{dA}{dx}-A(x)\frac{dB}{dx}}{B^2(x)}$

1.2.3 Implicit Differentiation

Implicit differentiation is used when you do not have an explicit solution for the dependent variable of interest. Here is an example:
Find dy/dx of the following function (it's the equation of a circle):

$$x^2 + y^2 = 1 \tag{1.2}$$

Differentiating each of the terms yields

$$2xdx + 2ydy = 0 \tag{1.3}$$

or

$$\frac{dy}{dx} = -x/y \tag{1.4}$$

A typical application would be the max/min problem, which could be accomplished by setting $dy/dx = 0$ which yields $x = 0$. This occurs when $y^2 = 1$ or $y = \pm 1$, as expected.

1.2.4 Logarithmic Differentiation

One would like to extend the product rule for more than two functions. This can be achieved with Logarithmic Differentiation. Suppose you want to find the derivative of

$$y(x) = A(x)B(x)C(x)D(x) \tag{1.5}$$

You could apply the product rule many times, or take the logarithm of both sides first:

$$ln(y(x)) = ln(A(x)) + ln(B(x)) + ln(C(x)) + ln(D(x)) \tag{1.6}$$

where a simple property of the logarithm has been used. Now, taking the derivative of both sides yields

$$\frac{1}{y(x)}\frac{dy}{dx} = \frac{1}{A(x)}\frac{dA}{dx} + \frac{1}{B(x)}\frac{dB}{dx} + \frac{1}{C(x)}\frac{dC}{dx} + \frac{1}{D(x)}\frac{dD}{dx} \tag{1.7}$$

Finally, multiplying both sides by $y(x)$ yields

$$\frac{dy}{dx} = B(x)C(x)D(x)\frac{dA}{dx} + A(x)C(x)D(x)\frac{dB}{dx} + A(x)B(x)D(x)\frac{dC}{dx} + A(x)B(x)C(x)\frac{dD}{dx} \tag{1.8}$$

Of course, if one is so inclined, one could generalize the results. If $y(x)$ is written

$$y(x) = \prod_{i=1}^{N} A_i(x) \tag{1.9}$$

then

$$\frac{dy}{dx} = \sum_{j=1}^{N} \frac{1}{A_j(x)} \frac{dA_j(x)}{dx} \prod_{i=1}^{N} A_i(x) \tag{1.10}$$

1.2.5 Pascal's Triangle

Consider Pascal's Triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Each of the numbers are obtained by adding the two adjacent numbers in the row above it. For example, 10 is below it's adjacent 4 and 6.

The rows of Pascal's Triangle represent the binomial coefficients, so that

$$(a+b)^3 = \mathbf{1} \cdot a^3 b^0 + \mathbf{3} \cdot a^2 b^1 + \mathbf{3} \cdot a^1 b^2 + \mathbf{1} \cdot a^0 b^3 \tag{1.11}$$

or more simply

$$(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3 \tag{1.12}$$

There is a formula for the binomial coefficients, given the "Choose Function":

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{1.13}$$

where n is the row and k goes from 1 to n. For example, the third number on the 5th row of Pascal's Triangle is a 10, but

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \cdot 4}{2 \cdot 1} = 10 \tag{1.14}$$

Note that the top solitary "1" is considered to be in the zeroth row and that $0! = 1$ by convention.

This relates to differential calculus, because there is a similar relationship that applies to the product rule to multiple derivatives. Suppose we are interested in the 4^{th} derivative of a product:

$$y^{(4)}(x) = (A(x)B(x))'''' \tag{1.15}$$

While the derivative terms that must exist are straightforward, it also happens that the coefficients of the derivatives are given by Pascal's Triangle, so that in this case

$$y^{(4)}(x) = A''''(x)B(x) + 4A'''(x)B'(x) + 6A''(x)B''(x) + 4A'(x)B'''(x) + A(x)B''''(x) \tag{1.16}$$

Thus, by inspection it can be seen that the 4^{th} derivative of xe^x is

$$\frac{d^4}{dx^4}(xe^x) = (4+x)e^x \tag{1.17}$$

as the first 3 terms are zero. By inspection, can you determine the 4^{th} derivative of x^2e^x? (See Bottom of Page for answer)[1]

The general case for the n^{th} derivative of a product of two functions $A(x)$ and $B(x)$ may be written

$$y^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} A^{(k)}(x)B^{(n)}(x) \tag{1.18}$$

It is left for the enthusiastic reader to obtain a general expression for the n^{th} derivative of the product of m functions.

1.3 Integral Calculus

Fast Facts:

1. Definition of a Definite Integral: $\int_a^b f(x)dx = lim_{N\to\infty}\left(\frac{b-a}{N}\right)\sum_0^{N-1} f\left(a + i\left(\frac{b-a}{N}\right)\right)$.
2. Definition of an Indefinite Integral: $\int f(x)dx = \int_a^x f(\hat{x})d\hat{x} + C$
3. The Indefinite Integral is an operator (it operates on functions).
4. In particular, the Indefinite Integral is the Accumulated Area Operator. The area is achieved by summing many rectangles of length $\Delta x = (b-a)/N$ and height $f(a + i\Delta x)$. Thus, the units of $\int_a^b f(x)dx$ is the units of $f(x)$ multiplied by the units of x. The integral introduces the peculiar-to-some idea of **Negative Area**. For example in integral calculus the area of a circle centered at the origin is NOT πr^2, it's ZERO as the bottom half of the circle is said to have negative area!
5. The Indefinite Integral of any Elementary Function **may or may not be** an Elementary Function.

1.3.1 Some Integrals

Function	Simple Form	Advanced Form
$x^n \quad (n \neq -1)$	$\frac{x^{n+1}}{n+1}$	$\int f^n(x)\frac{df}{dx}dx = \frac{f^{n+1}(x)}{n+1}$
x^{-1}	$ln\|x\|$	$\int \frac{df/dx}{f}dx = ln\|f(x)\|$
e^x	e^x	$\int e^{f(x)}\frac{df}{dx}dx = e^{f(x)}$
$ln(x)$	$xln(x) - x$	$\int ln(f(x))\frac{df}{dx}dx = f(x)ln(f(x)) - f(x)$
$sin(x)$	$-cos(x)$	$\int sin(f(x))\frac{df}{dx}dx = cos(f(x))$
$-cos(x)$	$-sin(x)$	
$-sin(x)$	$cos(x)$	
$cos(x)$	$sin(x)$	

1.3.2 Techniques of Integration

Technique	When to Use
u-Substitution	When it's obvious or when you're stuck.
Integration by Parts	When you have a product of two functions, and you know the derivative of one and the integral of the other.
Trigonometric Substitution	When you have $(a + x^2)$ or $(a - x^2)$ terms (especially in the denominator).
Synthetic Division/Partial Fraction	When you have a ratio of polynomials.
Series Solution	When you stuck but realize a Taylor Series is easy to calculate.

As you may recall, the formula for integration by parts is

$$\int udv = uv - \int vdu \tag{1.19}$$

[1] $(12 + 8x + x^2)e^x$

A common mistake when using integration by parts on a definite integral is to forget to evaluate the uv term with the integration limits.

1.3.3 Examples of Integration Techniques

Example (Advanced Forms)

$$\int \frac{x^2}{1+x^3}dx = \frac{1}{3}\int \frac{3x^2}{1+x^3}dx = \frac{1}{3}ln|1+x^3| + C \tag{1.20}$$

Comment: The "Advanced Forms" involve what I call the "Hope Method." You see a complicated function and hope that the derivative of the inside of the equation is sitting on top. In this case it is. One is sometimes taught to use u-Substitution here, but this integral should be done in your head. Notice how the integral would be much harder if it is $\int \frac{1}{1+x^3}dx$. That's why it's the Hope method, you hope that derivative is there!

Example (Advanced Forms)

$$\int x^5 ln(x)dx = \frac{1}{6}\int x^5 ln(x^6)dx = \frac{1}{36}\int 6x^5 ln(x^6)dx = \frac{1}{36}\left[x^6 ln|x^6| - x^6\right] = \frac{1}{36}\left[6x^6 ln|x| - x^6\right] + C \tag{1.21}$$

Comment: One of my favorite integrals. By using the properties of the logarithm, we can make the derivative show up by just multiplying by a constant. The integral could be done by integrating by parts, but it would be longer and much less cool!!

Example (Integration by Parts)

$$\int \underbrace{x}_{u}\underbrace{e^x dx}_{dv} = \underbrace{x}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\underbrace{dx}_{du} = e^x(x-1) + C \tag{1.22}$$

Comment: You can see that if we had $\int x^n e^x dx$ we would perform integration by parts n times, as the power of the monomial decreases by 1 every time, but the exponential stays firm. It is a good idea to take the derivative of your result to insure a correct answer. In this case the derivative is obtained by the product rule: $e^x(1) + (x-1)e^x = xe^x$ as it must.

Example (Synthetic Division)

$$\int \frac{x^3+2x^2+2x+1}{x+2}dx = \int x^2 + 2 - \frac{3}{x+2}dx = \frac{x^3}{3} + 2x - 3ln|x+2| + C \tag{1.23}$$

Comment: Use synthetic division on a rational function when the degree of the polynomial on top is greater than or equal to that on the bottom. Some students haven't done synthetic division in a while. It's just the same as long division you first learned in grade school. Unfortunately, you may have not done that in a while either!

Example (Partial Fraction Expansion)
The integrand of

$$\int \frac{1}{x^2+6x+8}dx \tag{1.24}$$

can be reduced by using a technique in ALGEBRA known as Partial Fraction Expansion. In particular, the integrand can be rewritten

$$\frac{1}{x^2+6x+8} = \frac{1}{(x+2)(x+4)} = \frac{A}{x+2} + \frac{B}{x+4} \tag{1.25}$$

Solving for A and B yields $A = 1/2$, $B = -1/2$. The solution is

$$\int \frac{1}{x^2+6x+8}dx = \frac{1}{2}[ln|x+2| - ln|x+4|] = \frac{1}{2}ln\left|\frac{x+2}{x+4}\right| + C \tag{1.26}$$

Comment: For the integral of a fifth degree polynomial divided by a second degree polynomial, one would use synthetic division first, and use Partial Fractions on the remaining integral. There is a technique where the partial fractions coefficients can be determined by inspection.

Example (Series Solution)
The following integral does not exist as a finite elementary function:

$$\int e^{-x^2}dx \tag{1.27}$$

However, we may proceed by noting that the Taylor Series expansion for an exponential function is

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \tag{1.28}$$

Thus, the integral becomes

$$\int e^{-x^2}dx = \int \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!}dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int x^{2n}dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{x^{2n+1}}{2n+1} \tag{1.29}$$

Comment: If you multiply our result by $2/\sqrt{\pi}$, we just found the Taylor Series expansion for erf(x).

Example (Trigonometric Substitution)
Find the "area" of a cirle. The equation of a circle is $x^2 + y^2 = r^2$. To find the area we will double the area of the top half of the circle, which can be found be integration:

$$A = 2\int_{-r}^{r}\sqrt{r^2-x^2}dx \tag{1.30}$$

Making the substitution $x = rcos(\phi)$, it follows that $dx = -rsin(\phi)d\phi$, and that

$$A = 2\int_{\pi}^{0}\sqrt{r^2-r^2cos^2(\phi)}(-rsin(\phi))d\phi = -2r^2\int_{\pi}^{0}sin^2(\phi))d\phi = -2r^2\int_{\pi}^{0}\left(\frac{1}{2}-\frac{1}{2}cos(2\phi)\right)d\phi = \pi r^2 \tag{1.31}$$

Comment: Trig sub gives expected answer.

Example (Bonus Fun - Coordinate System Conversion)

$$y_0 = \int_0^{\infty}e^{-x^2}dx = \int_0^{\infty}e^{-y^2}dy \tag{1.32}$$

Thus

$$y_0^2 = \int_0^{\infty}e^{-x^2}dx\int_0^{\infty}e^{-y^2}dy = \int_0^{\infty}\int_0^{\infty}e^{-x^2}e^{-y^2}dxdy \tag{1.33}$$

Using a rectangular to polar conversion yields

$$y_0^2 = \int_0^{\infty}\int_0^{\pi/2} e^{-r^2} r d\phi dr = \frac{\pi}{2}\left(-\frac{1}{2}\right)\int_0^{\infty} e^{-r^2}(-2r)dr = \frac{\pi}{4} \tag{1.34}$$

or

$$y_0 = \frac{\sqrt{\pi}}{2} \tag{1.35}$$

Comment: You can see why erf(x) has the $\frac{2}{\sqrt{\pi}}$ normalized factor in front - it's so that the area under the erf(x) is 1. This solution is a fun little trick that only works when the integral is extended to infinity. In general, though, if you see an integral whose limits are $-\infty$ to ∞, look to see if the integrand is an odd function (Hope Method). If it is, the integral is zero. Finally, notice that the area under the curve is finite, but that the length of the curve is infinite. This deserves some thought. If you had such a box, it would mean that you could fill the box with paint, but you would never have enough paint to paint the box!! Interestingly, It is also possible to have figures which have infinite perimeter with a finite area. Fractals have these properties.

1.4 Types and Methods of Solution

Since not all integrals have solutions in terms of Elementary functions, there are different types of solutions that may be obtained. Not all of these have equal merit.

1. Exact Analytical Explicit Solutions. (Best)
2. Exact Analytical Implicit Solutions. (Good)
3. Series Solutions. (Not as good)
4. Numerical Solutions. (OK, if nothing else works)

Students should familiarize themselves with the following solution methods.

1. Analytical Methods (such as the "Techniques of Integration", above).
2. Analytical and Numerical Solutions using Computer Algebra Systems (such as Maple).
3. Numerical Solutions using Spreadsheets.
4. Numerical Solutions using a programming language (such a C++).

As added motivation, students should be aware that

- These types and methods of solution apply to differential equations as well as integrals.
- Integrals are a special case of a differential equation.
- The laws of physics which apply to all of nature and devices created by man are governed by differential equations, **which, under many circumstances reduce to integrals**.
- Many of the algorithms for numerical integration and solution to differential equations are available (without charge) on the Internet (as is a host of related material). Try a web search for "Numerical Recipes in C++."

1.5 Applications of Calculus

There are very many applications of calculus. Below is a short table of functions and paramters of general interest to the mathematical and scientific community.

Application	Formula
Area under curve	$\int_a^b f(x)dx$
Slope of curve	$\frac{df}{dx}$
Extremum of a curve (max, min, inflection pt.)	$\frac{df}{dx} = 0$
Inflection point of a curve	$\frac{d^2f}{dx^2} = 0$
Arc Length	$\int_a^b \sqrt{1 + \left(\frac{df}{dx}\right)^2} dx$
Curvature (Radius)	$\frac{\left[1+\left(\frac{df}{dx}\right)^2\right]^{3/2}}{\left\lvert\frac{d^2f}{dx^2}\right\rvert}$
Average of a function	$\frac{1}{b-a}\int_a^b f(x)dx$
RMS of a function	$\sqrt{\frac{1}{b-a}\int_a^b f^2(x)dx}$
Center of Mass	$\int_a^b xf(x)dx$
Variance (Second Central Moment)	$\int_a^b (x-\bar{x})^2 f(x)dx$
Skewness (Third Central Moment)	$\int_a^b (x-\bar{x})^3 f(x)dx$
Kurtosis (Fourth Central Moment)	$\int_a^b (x-\bar{x})^4 f(x)dx$
Function Squared Norm	$\int_a^b \lvert f(x)\rvert^2 dx$
Fourier Transform	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$
Convolution Integral	$\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$
Taylor Coefficients	$\frac{1}{n!}\frac{d^nf}{dx^n}\rvert_{x=a}$
Fourier Sine Series	$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)sin(nt)dt$
Fourier Cosine Series	$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)cos(nt)dt$