Beginning Algebra Tutorial

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Tutorial 1

The Ideas of Algebra

Contents

1.1 Introduction

The goal of this tutorial is to show how algebra is merely a formal mathematical notation for what most people, even children, already know.

1.2 The Fundamental Rule of Algebra

Imagine two people, call them Person A and Person B, have the same amount of money.
If I take a dollar from each,

they still have the same amount of money.

If I double each of their cash amounts,

they still have the same amount of money.

If I steal 3 dollars from each of them,

they still have the same amount of money.

Regardless of what I do to them, as long as I do it to **both** of them, *they still have the same amount of money*!

This is not difficult to understand, even kids know this. Let's do a very simple example:

Example 1.0:

A basketball team has a 3-point specialist who only shoots 3-pointers. If he scored 12 points, how many 3-pointers did he make.

Solution 1.0:

I've asked kids this type of question before, and they always answer correctly: "4 3-pointers". Then I've asked them, "how did you do it"? and they respond with something like "easy, divide 12 by 3". "How did you know how that you should divide?" "I don't know." What they're doing is using algebra, even if they don't know that's what they're doing. From an algebra point of view, i.e. a written point of view, the problem is

$$3 \times (some\ number\ of\ baskets) = 12 \tag{1.1}$$

In algebra, we usually call an unknown quantity a "variable". Often we signify this variable as "x". So, the equation is

$$3x = 12 \tag{1.2}$$

To find x, we need to isolate it. We do this by dividing both sides of the equation by 3, which results in

$$x = 4 \tag{1.3}$$

Of course, we can only divide each side by 3 because of what I call the Fundamental Rule of Algebra. (Note: There is a "Fundamental Theorem of Algebra" which is very different). ♣

Example 1.1:

Jack wants to complete a 12 mile bike ride. He crashes after 8 miles. He gets back on his bike to start riding. How many miles does he have to go?

Solution 1.1:

In equation form, it looks like

$$8 + (some\ number\ of\ miles) = 12 \tag{1.4}$$

or

$$8 + x = 12 \tag{1.5}$$

Subtracting 8 from both sides yields

$$x = 4 \tag{1.6}$$

♣

From the Fundamental Rule of Algebra, we can add, subtract, multiply, or divide both sides by any number[1] Let's do a composite example with both addition and multiplication:

Example 1.2:

Solve the equation

$$\underbrace{3x + 5}_{PersonA} = \underbrace{14}_{PersonB} \tag{1.7}$$

Solution 1.2:

[1] Actually, adding, subtracting, multiplying, or dividing both sides by zero is not useful.

Think of this as two people who have the same amount of money. Person A has $3x + 5$ dollars and Person B has 14 dollars. To solve for the variable x, we need to "isolate" it one side of the equation. The simplest way to do this is to steal 5 dollars from each person! If we do that, then the equation becomes

$$3x = 9 \tag{1.8}$$

Now, dividing both sides by 3 yields the desired result

$$x = 3 \tag{1.9}$$

♣

At this level, all algebra is is formally writing down a set of rules that we already know!

Let's look at some more rules of algebra:

1.3 More Laws of Algebra

1.3.1 Commutative Law of Addition

Tom has no money. If you give him 5 dollars and then give him 10 dollars, then he has the same amount of money as if you would have given him 10 dollars and then given him 5 dollars. Duuuuhhhhhhhhh! Well, we actually have a name for this absolutely obvious result, and it's called the "Commutative Law of Addition". Mathematically, we write it like this

$$5 + 10 = 10 + 5 \tag{1.10}$$

or more generally,

$$a + b = b + a \tag{1.11}$$

Maybe we shouldn't give this law a name because it's so obvious, or perhaps call it the "Duh" Law, but in mathematics we choose to give it an intimidating name. For some this makes mathematics in general intimidating, smug, or incomprehensible. Though some naughty mathematicians enjoy you feeling this way, the real reason the name is used is because it is fairly precise.

1.3.2 Commutative Law of Multiplication

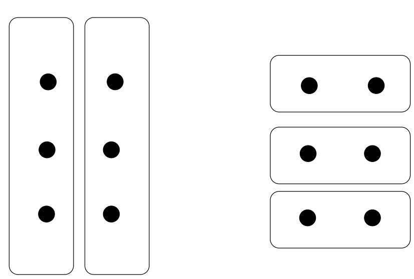
This one is a little less obvious, unless you think a little bit about what multiplication really is. It says that if you have 2 dollars and tripled your money, you would have the same if you would have 3 dollars and doubled your money. Mathematically,

$$2(3) = 3(2) \tag{1.12}$$

or more generally,

$$a(b) = b(a) \tag{1.13}$$

To make this result sensible, let's consider multiplication more closely. To do that, think about sets. If I doubled 3 dollars, then I'd have 2 sets of 3 dollars, as shown in the figure:



Tripling 2 dollars is having 3 sets of 2 dollars, as in the second part of the figure. Clearly, we have the same number of dollars either way.

If we wanted to count our pennies, we could put them into piles (or sets). If we had 6 piles of 8 pennies, we'd have 6(8) or 48 pennies. Indeed, this is why multiplication was created.

1.3.3 Associative Laws of Addition and Multiplication

Another obvious one. If you have three pockets with money in them, it wouldn't matter which order you counted them in. You could count the money in your 2 front pockets, add the total and then add it to the amount in your back pocket. Or, you could add the money in your back pocket to the money in one of your front pockets, then add that amount to your other front pocket. Either way, you have the same amount of money! Mathematically,

$$(a+b)+c = a+(b+c) \tag{1.14}$$

Similarly, the Associative Law of Multiplication is

$$(ab)c = a(bc) \tag{1.15}$$

1.3.4 Laws of Algebra Example

Example 1.3:

Solve

$$3x+2+5x+9 = 3x+4-x-2 \tag{1.16}$$

Solution 1.3:

First, combine like terms using the laws above

$$8x+11 = 2x+2 \tag{1.17}$$

subtracting $2x$ *from each side,*

$$6x+11 = 2 \tag{1.18}$$

subtracting 11 *from each side,*

$$6x = -9 \tag{1.19}$$

dividing 6 *from each side,*

$$x = -9/6 \tag{1.20}$$

Reducing the fraction (dividing top and bottom by 3) yields

$$x = -3/2 \tag{1.21}$$

♣

1.4 The Foil Method

Example 1.4:

Suppose you have two pockets, 4 Dollars in one and 5 Dollars in the other. Mystery Man A says he will triple the **total** *number of coins you have. Mystery Man B says that he will triple the coins in the first pocket and then triple the coins you have in the second. You can only choose one. Which one should you choose?*

Solution 1.4:

After a little thought, I'm sure you will agree that it does not matter, the total amount will be the same in either case. Mathematically, we would write this result as

$$3(4+5) = 3(4)+3(5) \tag{1.22}$$

In general, we would write

$$a(b+c) = a(b)+a(c) \tag{1.23}$$

♣

We could actually generalize the last result even more:

$$(a+b)(c+d) = ac + ad + bc + bd \tag{1.24}$$

I actually came close to inventing this myself in the 4^{th} grade. One day I had an odd thought - that 7×7 must be close to 6×8. Sure enough,

$$7 \times 7 = 49 \quad 6 \times 8 = 48 \quad \text{off by 1} \tag{1.25}$$

So, I continued in this way:

$$6 \times 6 = 36 \quad 5 \times 7 = 35 \quad \text{off by 1} \tag{1.26}$$

$$5 \times 5 = 25 \quad 4 \times 6 = 24 \quad \text{off by 1} \tag{1.27}$$

$$4 \times 4 = 16 \quad 3 \times 5 = 15 \quad \text{off by 1} \tag{1.28}$$

$$3 \times 3 = 9 \quad 2 \times 4 = 8 \quad \text{off by 1} \tag{1.29}$$

$$2 \times 2 = 4 \quad 1 \times 3 = 3 \quad \text{off by 1} \tag{1.30}$$

$$1 \times 1 = 1 \quad 0 \times 2 = 0 \quad \text{off by 1} \tag{1.31}$$

I knew that this was some new form of fancy math, bigger than what we were being taught. However, I couldn't proceed because I had never dreamed of the idea of representing numbers with letters. But, we can! In general, we would write

$$a \times a \;\; and \;\; (a-1) \times (a+1) \quad \text{must be off by 1} \tag{1.32}$$

so that

$$a^2 = (a-1)(a+1) + 1 \tag{1.33}$$

or more commonly

$$(a-1)(a+1) = a^2 - 1 \tag{1.34}$$

The reason math people don't talk about this much is because with the rules of algebra and the foil method, it is an absolutely obvious result:

$$(a-1)(a+1) = a^2 + a - a - 1 = a^2 - 1 \tag{1.35}$$

However, they should think about it! It has important ramifications for understanding geometry, physics, biology, chemistry, and the whole world around us! Here's how:

$$a^2 > (a+1)(a-1) \tag{1.36}$$

for any a. O.K., but you can also show that

$$a^2 > (a+2)(a-2) \tag{1.37}$$

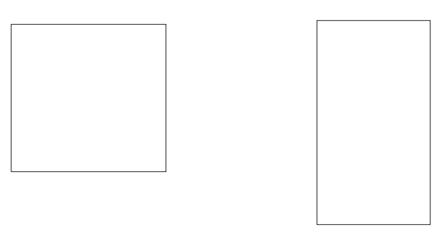
for any a, and that

$$a^2 > (a+3)(a-3) \tag{1.38}$$

and furthermore

$$a^2 > (a+b)(a-b) \tag{1.39}$$

for any b and any a. What this means is that if we have a piece of string (of fixed length) and want to make a rectangle with the largest area, we should make it a square. We can see this by drawing a picture:



The first is a square with four sides, each of length a. The area of the square is $a \times a$. The second is a rectangle, two sides of which are $a+1$ and two side of which are $a-1$. They both have the same perimeter, but the area of the square is larger. Put another way,

The Optimum Rectangle is the one with the most Symmetric Shape - the Square.

What's the most optimum shape? If we had a fixed perimeter and wanted to maximize the area and the shape didn't have to be a rectangle, what shape would it be? Congratulations, you guessed it:

The Most Optimum Shape is the Most Symmetric Shape - the Circle.

I saw an interesting film in a high school biology class. Some scientists wanted to determine, based on function, what the optimum shape would be for a Red Blood Cell. They programmed a computer to determine this, and they let it run for several weeks (would take just hours, now). What did the computer find? The optimum shape for a Red Blood Cell is the shape that Red Blood Cells actually have. They are Round!

Nature Likes Symmetry because Symmetry is Optimal.

The most common theory physicists have for how the universe works? "Super-Symmetry." These results are not surprising for someone who has thought about the ramifications of basic algebra!

1.5 Simultaneous Equations

Consider the equation

$$ax + b = c \tag{1.40}$$

This seems like a very hard equation, it looks like it has 4 unknowns. However, it is a mathematics convention that letters at the beginning of the alphabet (a,b,c) are "constants" which are considered to be known, and letters at the end of the alphabet (x, y, z) are considered to be unknowns which are called "variables". So, a mathematician would assume that only x in the equation is unknown, and immediately solve for it:

$$x = \frac{c-b}{a} \tag{1.41}$$

Of course, to do this one must assume that a is not zero.

What if you have two variables? Well, you use the following rule:

You must have the same number of INDEPENDENT equations as unknowns to obtain a solution.

So, if you have two variables you must have two equations.

Example 1.5:

Solve

$$3x + 4y = 3 \;\;,\;\; 3x - 4y = 4 \tag{1.42}$$

The basic procedure, called the "Substitution Method," to solve these equations would be to solve one of the equations for y, and substitute it into the other equation, which would then only have x's in it. Then isolate the x, and you have the x solution. You can plug you x solution into either of the original equations to get the y solution.

Here, however, it's just easier to add the equations, so that

$$6x = 7 \;\; or \;\; x = 7/6 \tag{1.43}$$

Plugging our x solution back into the first equation results in

$$3(7/6) + 4y = 3 \;\; or \;\; 7/2 + 4y = 3 \;\; or \;\; 4y = -1/2 \;\; or \;\; y = -1/8 \tag{1.44}$$

The reader should plug these x and y values into the each of the original equations to prove that they are satisfied.

OK, you get the idea, but what is an *independent* equation, or what are *dependent* equations? It's a slightly more advanced topic, and it's harder to prove the answer, but I'll give you an example of two dependent equations:

$$2x + 3y = 3 \ \ , \ \ 2x + 3y = 4 \tag{1.45}$$

If you pick x and y so that $2x + 3y = 3$, then obviously the second equation is not satisfied. There are **no solutions** to this problem.

Here's another example:

$$2x + 3y = 3 \ \ , \ \ 4x + 6y = 6 \tag{1.46}$$

If you multiply the first equation by 2, it is identical to the second equation. So, the second equation does not have any new information. There are **infinite solutions** to this set of equations.

1.6 Algebra's Dirty Little Secret - Transcendental Equations

I won't sugar coat it, **You've Been Lied To!** However, the lies are those of omission. It's not what you were told was wrong, it's that by not telling you about transcendental equations, you got the wrong impression. In particular,

Not all algebra problems are solvable.

Generally, you are given only ones that are. **People around the country (and the world) have this strange idea that math is just a series of procedures, like recipes - that it doesn't take any creativity. This is VERY WRONG**. In algebra it's fairly simple to see whether a problem is solvable or not. It's worse it calculus, because you can't just look at a problem and know whether it can be solved. You can look at a problem, waste a lot of time not being able to solve the problem, and conclude that the problem can't be done. (However, you could still be wrong!) In a sense, then, Integral Calculus is as much an art as it is a science. Back to algebra, consider the following equation:

$$\frac{2 * x}{1 + x * x} + 3x * x * x(1 - x) = 7 \tag{1.47}$$

There is no procedure to give the exact answer to that problem.[2]

It should be noted that there is an alternate notation for the above type of equation. In particular, $x * x * x$ is called x^3, so that the above equation is

$$\frac{2x}{1 + x^2} + 3x^3(1 - x) = 7 \tag{1.48}$$

Actually, x^n is a type of "function" called a polynomial....

1.7 What's Next

The next thing to do is to study different types of functions - get to know them one at a time. In particular, you should get to know the elementary functions: Sinusoidal functions ($sin(x)$, $cos(x)$, $tan(x)$), polynomial functions ($x^3 + 2x^2$), exponential functions (2^x, 3^x). You should know that any combinations of these elementary functions is an elementary function. You should know the graphs of each of these functions. You should know what inverse functions are, when they exist, and what they are for each of the elementary functions.

You should also learn basic methods in graphing. This certainly includes graphing terminology such as horizontal, vertical, and other asymptotes, concavity, monotonic, positive-definite, even/odd, bounded/unbounded, inflection points, local and global maxima and minima. You should get a clear picture of what $f(x + a)$ is, and how it differs from $f(x) + a$.

After you have attained some comfort level with algebra, then next subject to tackle is basic geometry, following by a more advanced form of geometry called trigonometry (which is just the geometry of triangles).

[2]However, one can come up with an approximate answer by making a series of guesses. Usually, a computer is employed to make guesses. However, there are procedures to improve upon a previous guess. So, the more guesses the computer makes, the closer to the solution it gets. This is extremely important in physics and engineering because the equation models a physical problem which they must have an answer.