Vector Algebra Tutorial

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Tutorial 1

Vector Algebra

Contents

1.1 Scalars and Vectors

	Data Types	Examples	Physical Quantities
Scalar	constants variables functions	$\sqrt{2}, 3, \pi$ x, y, t, T e^{-x^2}	time, Temperature
Vector	constants variables functions	$\sqrt{2}\mathbf{u_x} + 3\mathbf{u_y}$ $x\mathbf{u_x} + y\mathbf{u_y}$ $cos(\omega t)\mathbf{u_x} + sin(\omega t)\mathbf{u_y}$	velocity, Force

1.2 Rectangular and Polar Form

A 2-component vector can be represented by either a rectangular form (x, y), or a polar form (r, θ):

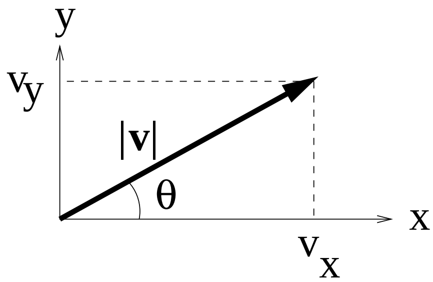

Figure 1.1: A 2D vector can be represented in rectangular or polar form.

In general, a 2D vector can be represented by

$$\mathbf{v} = v_x\mathbf{u_x} + v_y\mathbf{u_y} \tag{1.1}$$

From the figure, it can be seen that

$$v_x = |\mathbf{v}|\cos(\theta) \tag{1.2}$$
$$v_y = |\mathbf{v}|\sin(\theta) \tag{1.3}$$

which basically represent polar-to-rectangular conversions. Conversely, this equations can be inverted to obtain the corresponding rectangular-to-polar conversions:

$$\theta = tan^{-1}(v_y/v_x) \tag{1.4}$$
$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2} \tag{1.5}$$

When using the tan^{-1} function on your calculator, be sure the angle given back is in the correct quadrant. Also, the expression for the magnitude of a vector $|\mathbf{v}|$ can be generalized to n dimensions via the generalized Pythagorean Theorem:

$$|\mathbf{v}| = \sqrt{\sum_i^n v_i^2} \tag{1.6}$$

i.e. "the square root of the sum of the squares."

1.3 Direction Cosines

Combining Eq. foo and foo, it can be seen that a vector can be represented as

$$\mathbf{v} = |\mathbf{v}|[cos(\theta)\mathbf{u_x} + sin(\theta)\mathbf{u_y}] \tag{1.7}$$

or

$$\mathbf{v} = |\mathbf{v}|[cos(\theta)\mathbf{u_x} + cos(\pi/2 - \theta)\mathbf{u_y}] \tag{1.8}$$

where $\pi/2 - \theta$ is the angle of the vector from the y-axis.
We can generalize this concept for 3D vectors.

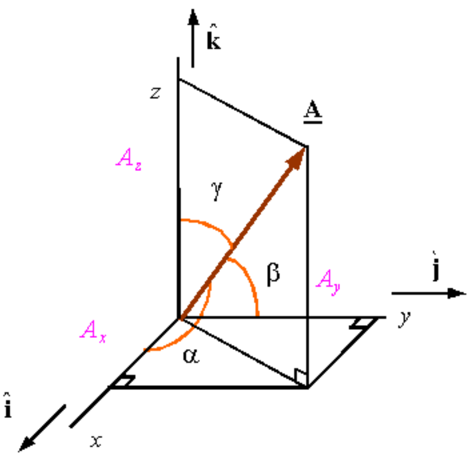

Figure 1.2: Direction Cosines

From the figure it can be seen that α is the angle from the x-axis to the vector, β is the angle from the y-axis to the vector, and γ is the angle from the z-axis to the vector, then the vector may be written

$$\mathbf{v} = |\mathbf{v}|[cos(\alpha)\mathbf{u_x} + cos(\beta)\mathbf{u_y} + cos(\gamma)\mathbf{u_z}] \tag{1.9}$$

Of course, $\mathbf{v}/|\mathbf{v}|$ is the unit vector in the direction of $\mathbf{v}$, so that

$$\mathbf{u_v} = cos(\alpha)\mathbf{u_x} + cos(\beta)\mathbf{u_y} + cos(\gamma)\mathbf{u_z} \tag{1.10}$$

These are known as the "direction cosines" of the vector. Solving for the angles yields

$$\alpha = cos^{-1}\left(\frac{v_x}{\sqrt{v_x^2 + v_y^2 + v_z^2}}\right) \tag{1.11}$$

$$\beta = cos^{-1}\left(\frac{v_y}{\sqrt{v_x^2 + v_y^2 + v_z^2}}\right) \tag{1.12}$$

$$\gamma = cos^{-1}\left(\frac{v_z}{\sqrt{v_x^2 + v_y^2 + v_z^2}}\right) \tag{1.13}$$

1.4 Vector Operations

In general, we will consider 3D vectors, though much of the results apply to 2D vectors as well.

1.4.1 Vector Addition

If we define

$$\mathbf{A} = A_x \mathbf{u_x} + A_y \mathbf{u_y} + A_z \mathbf{u_z} \tag{1.14}$$

$$\mathbf{B} = B_x \mathbf{u_x} + B_y \mathbf{u_y} + B_z \mathbf{u_z} \tag{1.15}$$

then there is precisely one way to define vector addition:

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{u_x} + (A_y + B_y)\mathbf{u_y} + (A_z + B_z)\mathbf{u_z} \tag{1.16}$$

1.4.2 Multiplication of a Scalar and a Vector

Similarly, there is only one way to define the multiplication of a scalar and a vector:

$$a\mathbf{A} = (aA_x)\mathbf{u_x} + (aA_y)\mathbf{u_y} + (aA_z)\mathbf{u_z} \tag{1.17}$$

1.4.3 Vector Multiplication

However, there are many ways to define the product of two vectors. In general, vector multiplication must take the form

$$\begin{aligned}
\mathbf{A}(\mathbf{B}) &= (A_x \mathbf{u_x} + A_y \mathbf{u_y} + A_z \mathbf{u_z})(B_x \mathbf{u_x} + B_y \mathbf{u_y} + B_z \mathbf{u_z}) \\
&= A_x B_x \mathbf{u_x}(\mathbf{u_x}) + A_x B_y \mathbf{u_x}(\mathbf{u_y}) + A_x B_z \mathbf{u_x}(\mathbf{u_z}) \\
&+ A_y B_x \mathbf{u_y}(\mathbf{u_x}) + A_y B_y \mathbf{u_y}(\mathbf{u_y}) + A_y B_z \mathbf{u_y}(\mathbf{u_z}) \\
&+ A_z B_x \mathbf{u_z}(\mathbf{u_x}) + A_z B_y \mathbf{u_z}(\mathbf{u_y}) + A_z B_z \mathbf{u_z}(\mathbf{u_z})
\end{aligned} \tag{1.18}$$

Again, there are many possible definitions for the product of two given unit vectors. For example, should the result be a scalar or a vector? (And if a vector, what dimension?)

Dot Product

The dot product of two vectors results in a scalar, and thus the dot product is called the "scalar product." The "rule" that governs the dot product is

$$\mathbf{u_a} \cdot \mathbf{u_b} = \delta_{ab} \equiv \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases} \tag{1.19}$$

The function δ_{ab} is called the Kronecker delta function. It follows that only the diagonal terms in Eq. (1.18) survive and that

$$\boxed{\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z} \tag{1.20}$$

It follows that

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2 \tag{1.21}$$

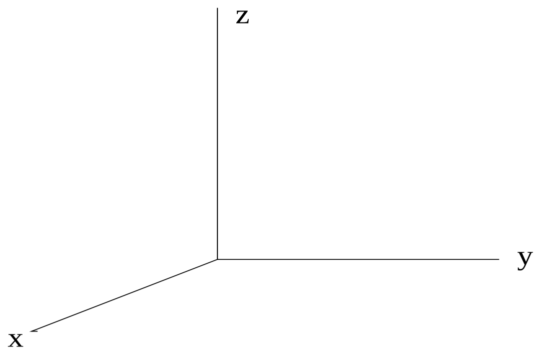

Figure 1.3: A right handed Cartesian coordinate system is defined so that $\mathbf{u_x} \times \mathbf{u_y} = \mathbf{u_z}$. It is best to not use left-handed coordinate systems.

Cross Product

The cross product of two vectors results in a vector, and thus the cross product is called the "vector product." The "rule" that governs the vector product is the right hand rule. Using the figure, and starting with your right hand with your palm flat, point your fingers toward the x-axis. Then curl your fingers (except your thumb) around toward the y-axis making a closed fist. Your thumb should be pointing in the z-direction.

The operation is

$$\mathbf{u_x} \times \mathbf{u_y} = \mathbf{u_z} \tag{1.22}$$

Using this procedure, you should also be able to confirm (Do it now!) that

$$\mathbf{u_x} \times \mathbf{u_z} = -\mathbf{u_y} \tag{1.23}$$

$$\mathbf{u_y} \times \mathbf{u_x} = -\mathbf{u_z} \tag{1.24}$$

$$\mathbf{u_y} \times \mathbf{u_z} = \mathbf{u_x} \tag{1.25}$$

$$\mathbf{u_z} \times \mathbf{u_x} = \mathbf{u_y} \tag{1.26}$$

$$\mathbf{u_z} \times \mathbf{u_y} = -\mathbf{u_x} \tag{1.27}$$

and it is conventional to define

$$\mathbf{u_a} \times \mathbf{u_a} = \mathbf{0} \tag{1.28}$$

for any unit vector $\mathbf{u_a}$. It follows that for any vector, $\mathbf{A}$,

$$\mathbf{A} \times \mathbf{A} = \mathbf{0} \tag{1.29}$$

Beside the right hand rule definition, there is another memory device for those without right hands! Applying these results to Eq. (1.18) yields

$$\mathbf{A} \times \mathbf{B} = +\mathbf{u_x}(A_yB_z - A_zB_y) - \mathbf{u_y}(A_xB_z - A_zB_x) + \mathbf{u_z}(A_xB_y - A_yB_x) \tag{1.30}$$

which can be rewritten in matrix form as

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{u_x} & \mathbf{u_y} & \mathbf{u_z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{1.31}$$

Question: Does this same result apply if we use a left-hand rule? What if we use a left-handed coordinate system? What *is* a left-handed coordinate system?

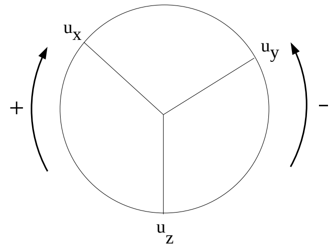

Figure 1.4: Go around the circle in the correct direction to obtain the cross product.

1.5 Two Theorems

It can be shown that

$$\boxed{\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|cos(\theta_{ab})} \tag{1.32}$$

$$\boxed{\mathbf{A}\times\mathbf{B} = |\mathbf{A}||\mathbf{B}|sin(\theta_{ab})\mathbf{u_{ab}}} \tag{1.33}$$

where θ_{ab} is the angle between the vectors $\mathbf{A}$ and $\mathbf{B}$. The unit vector $\mathbf{u_{ab}}$ is in the direction perpendicular to $\mathbf{A}$ and $\mathbf{B}$ in accordance with the right hand rule.

Example 1.0:

Find a unit vector normal to $\mathbf{A}$ *and* $\mathbf{B}$*, where* $\mathbf{A} = (1, 2, 3)$ *and* $\mathbf{B} = (4, 5, 6)$.

Solution 1.0:

The vector

$$\mathbf{v} = \mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{u_x} & \mathbf{u_y} & \mathbf{u_z} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = -3\mathbf{u_x} + 6\mathbf{u_y} - 3\mathbf{u_z}$$

However, $\mathbf{v}$ *is not a unit vector. It can be made a unit vector by dividing it by its magnitude:*

$$\mathbf{u_{ab}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-\mathbf{u_x} + 2\mathbf{u_y} - \mathbf{u_z}}{\sqrt{6}}$$

♣

Another common problem is to find the angle between two vectors. Since both of the product theorem have the angle in them, it follows that

$$\theta_{ab} = cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right) = sin^{-1}\left(\frac{|\mathbf{A}\times\mathbf{B}|}{|\mathbf{A}||\mathbf{B}|}\right) \tag{1.34}$$

Example 1.1:

Find the angle between $\mathbf{A}$ *and* $\mathbf{B}$ *in the previous example.*

Solution 1.1:

$$\theta_{ab} = cos^{-1}\left(\frac{1(4) + 2(5) + 3(6)}{\sqrt{1^2 + 2^2 + 3^2}\sqrt{4^2 + 5^2 + 6^2}}\right) = cos^{-1}\left(\frac{32}{7\sqrt{22}}\right) = 12.9^\circ \tag{1.35}$$

Students should confirm the answer via the cross product. ♣

1.6 Physical Interpretation of the Products

1.6.1 Dot Product

If we consider two vectors **A** and **B**, neither of which are the zero vector, then

$$\mathbf{A} \cdot \mathbf{B} = 0 \Leftrightarrow \mathbf{A} \perp \mathbf{B} \tag{1.36}$$

In words, the dot product of perpendicular vectors is zero.

The (magnitude of the) dot product can be interpreted by noting that the projection of **A** onto **B** is

$$|\mathbf{A}| cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B}|} \tag{1.37}$$

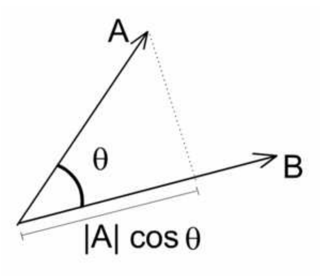

Figure 1.5: The magnitude of the dot product is proportional to the projection of A onto B (and vice versa).

1.6.2 Cross Product

If we consider two nonzero vectors, then it can also be seen that

$$\mathbf{A} \times \mathbf{B} = \mathbf{0} \Leftrightarrow \mathbf{A} \parallel \mathbf{B} \tag{1.38}$$

In words, the cross product of parallel vectors is zero.

$$\mathbf{A} \times \mathbf{B} = \mathbf{C} \neq \mathbf{0} \Leftrightarrow \mathbf{C} \perp \mathbf{A}, \mathbf{B} \tag{1.39}$$

In words, the result of a cross product of any two vectors is always perpendicular to the two vectors. This can be used to obtain a new vector perpendicular to any to vectors.

We now know the direction of the cross product, what does the magnitude represent physically? The magnitude represents an area as shown the the figure.

Example 1.2:

If the path of a particle is governed by

$$\mathbf{r(t)} = bsin(\omega t)\mathbf{u_x} + bcos(\omega t)\mathbf{u_y} + ct^2\mathbf{u_z} \tag{1.40}$$

then what is the magitude of the acceleration of the particle?

Solution 1.2:

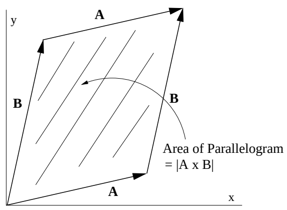

Figure 1.6: A right handed Cartesian coordinate system is defined so that $\mathbf{u_x} \times \mathbf{u_y} = \mathbf{u_z}$.

Taking derivatives, yields

$$\mathbf{v(t)} = \omega b cos(\omega t)\mathbf{u_x} - \omega b sin(\omega t)\mathbf{u_y} + 2ct\mathbf{u_z} \tag{1.41}$$

$$\mathbf{a(t)} = \omega^2 b sin(\omega t)\mathbf{u_x} + \omega^2 b cos(\omega t)\mathbf{u_y} + 2c\mathbf{u_z} \tag{1.42}$$

Finally, the magitude of the acceleration is

$$|\mathbf{a(t)}| = \sqrt{\omega^4 b^2 + 4c^2} \tag{1.43}$$

The student should

- *convince themself that the bounding shape of the motion is not conical motion,*
- *determine the bounding shape of the motion.*

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1.7 Right-Handed Screws

Most screws (and things that screw in) are right-handed, thus

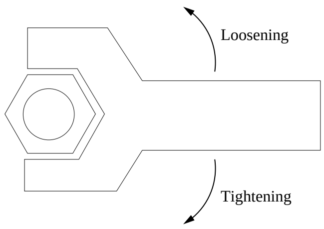

Figure 1.7: Righty-Tighty, Lefty-Loosy.

1.8 Vectors in Other Coordinate Systems

The student should fill in the following figure:

Using the results, it can be seen that

$$\mathbf{u_r} = cos\phi\mathbf{u_x} + sin\phi\mathbf{u_y} \tag{1.44}$$

Fill in the following blank:

$$\mathbf{u_\phi} = \_\_\_\_\_\_\_\_\_\_\_\_\mathbf{u_x} + \_\_\_\_\_\_\_\_\_\_\_\_\mathbf{u_y} \tag{1.45}$$

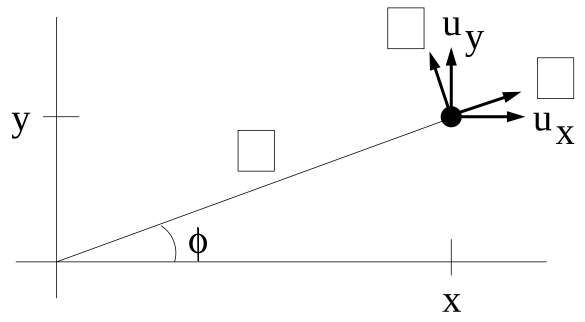

Figure 1.8: Fill in the boxes for polar coordinates.

1.9 More Vector Algebra Theorems

The famous "BAC-CAB" identity is

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{1.46}$$

and the "Scalar Triple Product" is

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) \tag{1.47}$$

$$= \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \tag{1.48}$$

$$= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \tag{1.49}$$

Additional theorems can be found using a web search for "Vector algebra theorems", or, more specifically

- Vector Quadruple Product
- Vector Triple Product
- Lagrange's Identity

1.10 Problems and Exercises

P1.0. Wow!